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BOUNDARY OBSERVABILITY FOR THE FINITE-DIFFERENCE SPACE SEMI-DISCRETIZATIONS OF THE 2-D WAVE EQUATION IN THE SQUARE

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ABSTRACT. – We extend our previous results on the boundary observability of the finite-difference space semidiscretizations of the 1-d wave equation to 2-d in the square. As in the 1-d case, we prove that the constants on the boundary observability inequality blow-up as the mesh-size tends to zero. However, we prove a uniform observability inequality in a subspace of solutions generated by the low frequencies. The dimension of these subspaces grows as the mesh size tends to zero and eventually, in the limit, covers the whole energy space. Our result is sharp in the sense that the uniformity of the observability inequality is lost when the dimension of the subspaces grows faster. Our method of proof combines discrete multiplier techniques, Fourier series developments and compactness-uniqueness arguments. © Elsevier, Paris

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RÉSUMÉ. – On considère les semi-discrétisations en espace par différences finies de l'équation des ondes 2-d dans un carré. On étudie l'observabilité frontière uniforme lorsque le pas de la discrétisation tend vers zéro. On montre que, à cause des hautes fréquences, l'observabilité uniforme n'a pas lieu. On établiet ensuite des inégalités d'observabilité uniforme dans des sous-espaces de solutions où les hautes fréquences ont été tronquées ou filtrées. Le résultat est optimal en ce qui concerne le taux de croissance de ces sous-espaces lorsque le pas de la discrétisation tend vers zéro. La méthode de démonstration combine des techniques de multiplicateurs discrets, des développements de Fourier et des arguments de compacité-unicité. © Elsevier, Paris

1. Introduction

Let Ω be the square $\Omega = (0,\pi)\times(0,\pi)$ of $\mathbb{R}^2$ and consider the wave equation with Dirichlet boundary conditions:

$$
(1.1)\qquad \begin{cases} u'' - \Delta u = 0 & \text{in } Q = \Omega \times (0,T), \\ u = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ u(x,0) = u^0(x),\ u'(x,0) = u^1(x) & \text{in } \Omega. \end{cases}
$$

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In (1.1) $' = \partial/\partial t$ denotes partial derivation with respect to time and Δ is the Laplacian in the space variable $x = (x_1, x_2) \in \Omega$.

Given $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ system (1.1) admits a unique solution

$$u \in C\big([0,T]; H_0^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big).$$

Moreover, the energy

$$E(t) = \frac{1}{2} \int_{\Omega} \big[|u_t(x,t)|^2 + |\nabla u(x,t)|^2\big]\,\mathrm{d}x \tag{1.2}$$

remains constant, i.e.,

$$E(t) = E(0), \quad \forall 0 < t < T. \tag{1.3}$$

Let Γ_0 denote a subset of the boundary of Ω constituted by two consecutive sides, for instance,

$$\Gamma_0 = \big\{(x_1, \pi)\colon\ x_1 \in (0,\pi)\big\} \cup \big\{(\pi, x_2)\colon\ x_2 \in (0,\pi)\big\}. \tag{1.4}$$

It is by now well-known (see [14]) that for $T > 2\sqrt{2}\pi$ there exists $C(T) > 0$ such that

$$E(0) \leqslant C(T) \int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t \tag{1.5}$$

holds for every finite-energy solution of (1.1).

In (1.5), n denotes the outward unit normal to Ω, $\partial \cdot /\partial n$ the normal derivative and $\mathrm{d}\sigma$ the surface measure.

Remark 1.1. – (a) The lower bound $2\sqrt{2}\pi$ on the minimal observability time is sharp. On the other hand inequality (1.5) fails if in the right hand side of (1.5) instead of Γ_0 we only consider the energy concentrated on a strict subset of Γ_0. These two facts can be proved with the aid of the Gaussian beams solutions by J. Ralston [15] as in [2].

(b) We refer to [2,3] and [4] for sharp sufficient conditions in terms of geometric optics for the boundary observability of the wave equation in smooth domains.

The goal of this paper is to analyze the boundary observability of some semi-discrete approximations in space of the wave equation (1.1).

Let us consider the finite-difference semi-discretization of (1.1). Given $J, K \in \mathbb{N}$ we set

$$h_1 = \frac{\pi}{J+1}, \qquad h_2 = \frac{\pi}{K+1}. \tag{1.6}$$

We denote by $u_{j,k}(t)$ the approximation of the solution u of (1.1) at the point $x_{j,k} = (jh_1, kh_2)$. The finite-difference semi-discretization of (1.1) is as follows:

$$\begin{cases} u''_{jk} - \dfrac{u_{j+1,k} + u_{j-1,k} - 2u_{j,k}}{h_1^2} - \dfrac{u_{j,k+1} + u_{j,k-1} - 2u_{j,k}}{h_2^2} = 0 \\ \quad 0 < t < T,\ j = 1, \ldots, J;\ k = 1, \ldots, K, \\ u_{j,k} = 0, \quad 0 < t < T,\ j = 0,\ J+1;\ k = 0, K+1, \\ u_{j,k}(0) = u^0_{j,k},\ u'_{j,k}(0) = u^1_{j,k}, \quad j = 1, \ldots, J;\ k = 1, \ldots, K. \end{cases} \tag{1.7}$$

In (1.7), the first equation provides a 5-point approximation of the wave equation. The second equation takes account of the homogeneous Dirichlet boundary conditions. The last one provides the initial conditions guaranteeing the uniqueness of the solution. System (1.7) is a coupled system of JK linear ordinary differential equations of second order.

Let us now introduce the *discrete energy* associated with system (1.7):

$$E_{h_1,h_2}(t) = \frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \left[\left| u'_{jk}(t) \right|^2 + \left| \frac{u_{j+1,k}(t) - u_{j,k}(t)}{h_1} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{j,k}(t)}{h_2} \right|^2 \right]. \tag{1.8}$$

It is easy to see that the energy remains constant in time, i.e.,

$$E_{h_1,h_2}(t) = E_{h_1,h_2}(0), \quad \forall 0 < t < T \tag{1.9}$$

for every solution of (1.7).

We now observe that the discrete version of the energy observed on the boundary (i.e., of $\int_0^T \int_{\Gamma_0} |\partial u/\partial n|^2 \, \mathrm{d}\sigma \, \mathrm{d}t$) is given by:

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 \mathrm{d}\sigma \, \mathrm{d}t \sim \int_0^T \left[h_1 \sum_{j=1}^{J} \left| \frac{u_{j,K}(t)}{h_2} \right|^2 + h_2 \sum_{k=1}^{K} \left| \frac{u_{J,k}(t)}{h_1} \right|^2 \right] \mathrm{d}t. \tag{1.10}$$

Indeed, $|u_{j,K}/h_2|^2$ is the most natural approximation of $|\partial u/\partial n|^2$ at the point $x = x_{j,K+1} = (jh_1, \pi)$. We refer to Remark 4.5 below for a detailed justification of this choice for the discrete normal derivative.

The discrete version of (1.5) is then an inequality of the form

$$E_{h_1,h_2}(0) \leqslant C_{h_1,h_2}(T) \int_0^T \left[h_1 \sum_{j=1}^{J} \left| \frac{u_{j,K}(t)}{h_2} \right|^2 + h_2 \sum_{k=1}^{K} \left| \frac{u_{J,k}(t)}{h_1} \right|^2 \right] \mathrm{d}t. \tag{1.11}$$

As we shall see, (1.11) holds for any $T > 0$ and any $h_1, h_2 > 0$ as in (1.6), for a suitable constant $C_{h_1,h_2}(T) > 0$.

The problem we discuss here can be formulated as follows: *Assuming $T > 2\sqrt{2}\pi$, is the constant $C_{h_1,h_2}(T)$ in* (1.11) *uniformly bounded as $h_1, h_2 \to 0$? Or, in other words, can we recover the observability inequality* (1.5) *as the limit as $h_1, h_2 \to 0$ of the inequalities* (1.11) *for the semi-discrete systems* (1.7)?

This problem is motivated by the numerical implementation of the boundary controllability property of the wave equation (see [1,5–7]).

As it was already observed in [5], the constants $C_{h_1,h_2}(T)$ in (1.11) necessarily blow-up as $h_1, h_2 \to 0$. This is due to the fact that spurious high frequency oscillations are present in the semi-discrete system (1.7). This result may be rigorously stated as follows:

THEOREM 1.1. – *For any $T > 0$ we have*

$$\sup_{u \text{ solution of } (1.7)} E_{h_1,h_2}(0)\left(\int_0^T \left[h_1 \sum_{j=1}^{J}\left|\frac{u_{j,K}(t)}{h_2}\right|^2 + h_2 \sum_{k=1}^{K}\left|\frac{u_{J,k}}{h_1}\right|^2\right] \mathrm{d}t\right)^{-1} \to \infty \tag{1.12}$$

as $h_1, h_2 \to 0$.

This result will be proved in Section 2 through the spectral analysis of system (1.7).

In order to prove the positive counterpart of Theorem 1.1 we have to filter or truncate the high frequencies. To do that we consider the eigenvalue problem associated with (1.7):

$$\begin{cases} -\dfrac{\varphi_{j+1,k} + \varphi_{j-1,k} - 2\varphi_{j,k}}{h_1^2} - \dfrac{\varphi_{j,k+1} + \varphi_{j,k-1} - 2\varphi_{j,k}}{h_2^2} = \lambda \varphi_{j,k} \\ \quad j = 1, \ldots, J;\ k = 1, \ldots, K, \\ \varphi_{j,k} = 0, \quad j = 0, J+1;\ k = 0, K+1. \end{cases} \tag{1.13}$$

System (1.13) admits JK eigenvalues. The following is a sharp upper bound for the eigenvalues of (1.13):

$$\lambda \leqslant 4\left[\frac{1}{h_1^2} + \frac{1}{h_2^2}\right]. \tag{1.14}$$

As we shall see in Section 2, the blow-up of the observability constant (1.12) is due to solutions of (1.7) of the form $u = e^{\sqrt{\lambda t}}\varphi$, λ being sufficiently large eigenvalues of (1.13) and φ the corresponding eigenfunctions. Indeed, as we shall see, the high frequency eigenfunctions of system (1.13) are such that the energy concentrated on the boundary is asymptotically smaller than the total energy.

In order to get uniform observability estimates we first observe that solutions of (1.7) can be developed in Fourier series of the form:

$$u = \sum_{\lambda \text{ e.v. of } (1.13)} \left[a_\lambda^+ e^{i\sqrt{\lambda}t} + a_\lambda^- e^{-i\sqrt{\lambda}t}\right]\varphi_\lambda \tag{1.15}$$

where the sum runs over al eigenvalues of (1.13), $a_\lambda^\pm$ are complex coefficients and φ_λ are the eigenvectors of (1.13).

It seems natural to introduce the following classes of solutions of (1.7) in which the high frequencies have been truncated or filtered.

For any $0 < \gamma \leqslant 4$ we set

$$\begin{aligned} &\mathcal{C}_\gamma(h_1, h_2) \\ &\quad = \left\{u \text{ solution of (1.7) of the form } u = \sum_{\lambda \leqslant \gamma[h_1^{-2} + h_2^{-2}]} \left[a_\lambda^+ e^{i\sqrt{\lambda}t} + a_\lambda^- e^{-i\sqrt{\lambda_k}t}\right]\varphi_\lambda\right\}. \end{aligned} \tag{1.16}$$

Note that, according to the upper bound (1.14), when $\gamma = 4$, $\mathcal{C}_\gamma(h_1, h_2) = \mathcal{C}_4(h_1, h_2)$ coincides with the space of all solutions of (1.16). However, when $0 < \gamma < 4$, solutions in the class

$\mathcal{C}_\gamma(h_1,h_2)$ do not contain the contribution of the high frequencies $\lambda > \gamma(h_1^{-2}+h_2^{-2})$ that have been truncated or filtered.

The following result asserts that, whatever $0<\gamma<4$ is, the uniform observability does not hold in the classes $\mathcal{C}_\gamma(h_1,h_2)$.

THEOREM 1.2. – *For any $T>0$ and $0<\gamma\leqslant 4$, there exist sequences $h_1,h_2\to 0$ such that*

$$\sup_{u\in\mathcal{C}_\gamma(h_1,h_2)} E_{h_1,h_2}(0)\left(\int_0^T\left[h_1\sum_{j=1}^{J}\left|\frac{u_{j,K}(t)}{h_2}\right|^2+h_2\sum_{k=1}^{k}\left|\frac{u_{J,k}}{h_1}\right|^2\right]\mathrm{d}t\right)^{-1}\to\infty. \tag{1.17}$$

Remark 1.2. – Let us compare Theorem 1.2 with the 1-d results in [8,9]. In 1-d there is one single parameter for the mesh size. Let us denote it by $h>0$. The 1-d upper bound for the spectrum is then $\lambda\leqslant 4h^{-2}$. As it was proved in [8,9], due to spurious high frequency vibrations the observability constant blows up as $h\to 0$ in 1-d too. However, in [8,9] it was shown that if $0<\gamma<4$, in the class $\mathcal{C}_\gamma(h)$ of solutions of the semi-discrete wave equation in which the Fourier components vanish for $\lambda\geqslant\gamma h^{-2}$, the observability constant remains bounded as $h\to 0$ for $T>0$ large enough.

Theorem 1.2 shows that the 2-d analogue is not true. This is due to the fact that, even when $\lambda\leqslant\gamma(h_1^{-2}+h_2^{-2})$ with $0<\gamma<4$, the eigenfunctions may present spurious oscillations in some space direction for high frequencies.

Note however that Theorem 1.2 guarantees the blow-up of the observability constant for particular sequences $h_1,h_2\to 0$ and that it does not exclude the existence of other sequences $h_1,h_2\to 0$ for which the supremum in (1.17) remains bounded.

Actually, our proof of Theorem 1.2 requires that $h_1,h_2\to 0$ so that

$$\sup|h_2/h_1|<\sqrt{\gamma/(4-\gamma)},$$

or, by symmetry,

$$\sup|h_1/h_2|<\sqrt{\gamma/(4-\gamma)}.$$

As we shall see in Section 3, the result is sharp since, for instance, when $h_1=h_2=h$, the uniform observability holds in the class $\mathcal{C}_\gamma(h_1,h_2)$ as soon as $\gamma<2$. This indicates that getting sharp sufficient conditions for the uniform observability requires not only cutting-of the high frequencies but also choosing the ratio between the mesh-parameters in an appropriate way.

The positive counterpart of Theorem 1.1 and 1.2 will be stated and proved in Section 3 since the description of the appropriate filtering of high frequencies requires a precise analysis of the spectrum of the system.

All the results of this paper can be easily extended to te following cases:

(a) Ω is a rectangle of $\mathbb{R}^2$;

(b) Ω is a hypercube of $\mathbb{R}^n$ with $n\geqslant 2$ or even $\Omega=\prod_{i=1}^n(a_i,b_i)$.

The rest of the paper is organized as follows. In Section 2 we analyze the spectrum of the discrete system and prove the negative results of Theorems 1.1 and 1.2. In Section 3 we prove positive results guaranteeing the uniform observability once the high frequencies are cut off. In Section 4 we improve the observability result of Section 3 obtaining a better control time. This is done by an adaptation of a compactness-uniqueness argument, often used in the PDE context, to the present discrete case.

2. Spectral analysis: Non-uniform observability

The eigenvalues and eigenvectors of system (1.13) may be computed explicitly (see [11, p. 459]).

The eigenvalues of system (1.13) are as follows:

$$\lambda^{p,q}(h_1,h_2)=4\left[\frac{1}{h_1^2}\sin^2\left(\frac{ph_1}{2}\right)+\frac{1}{h_2^2}\sin^2\left(\frac{qh_2}{2}\right)\right],\quad p=1,\dots,J;\ q=1,\dots,K, \tag{2.1}$$

and the corresponding eigenvectors:

$$\varphi^{p,q}=\left(\varphi_{j,k}^{p,q}\right)_{\substack{1\leqslant j\leqslant J\\ 1\leqslant k\leqslant K}},\quad \varphi_{j,k}^{p,q}=\sin(jph_1)\sin(kqh_2). \tag{2.2}$$

Let us also recall what the spectrum of the continuous system is. The eigenvalue problem associated with (1.1) is

$$-\Delta\varphi=\lambda\varphi \quad \text{in } \Omega;\quad \varphi=0 \quad \text{on } \partial\Omega. \tag{2.3}$$

The eigenvalues of the continuous problem are

$$\lambda^{p,q}=p^2+q^2 \tag{2.4}$$

and the corresponding eigenfunctions

$$\varphi^{p,q}(x_1,x_2)=\sin(px_1)\sin(qx_2). \tag{2.5}$$

Observe that the eigenvectors of the discrete problem coincide with the eigenfunctions of the continuous one at the mesh points x_{jk}. Roughly speaking, one can say that the eigenvectors of the discrete problems and the eigenfunctions of the continuous one are the same. Therefore, as in the continuous case, the eigenvectors of the discrete problem are given in separated variables as a product of a function of x_1 times a function of x_2. Moreover, each of these functions is an eigenvector of a 1-d discrete problem. The following properties are easy to check:

PROPOSITION 2.1. – *The following properties hold*:
(a) $\lambda^{p,q}(h_1,h_2)\to\lambda^{p,q}$ *as* $h_1,h_2\to 0$ *for all* $p,q\in\mathbb{N}$.
(b) $\lambda^{p,q}(h_1,h_2)\leqslant\lambda^{p,q},\ \forall(p,q),\ \forall h_1,h_2>0$.
(c) $\lambda^{p,q}(h_1,h_2)\leqslant 4[h_1^{-2}+h_2^{-2}],\ \forall(p,q),\ \forall h_1,h_2>0$.
(d) $\lambda^{p,q}(h_1,h_2)/(h_1^{-2}+h_2^{-2})\to 4$ *for* $p=J,\ q=K$ *as* $h_1,h_2\to\infty$.
(e) *For* (p,q) *fixed,*

$$\begin{cases}\lambda^{p,q}(h_1,h_2)\to\lambda^{p,q}(0,h_2)=4\left[\dfrac{p^2}{4}+\dfrac{1}{h_2^2}\sin^2\left(\dfrac{qh_2}{2}\right)\right] & \textit{as } h_1\to 0,\\[2ex] \lambda^{p,q}(h_1,h_2)\to\lambda^{p,q}(h_1,0)=4\left[\dfrac{1}{h_1^2}\sin^2\left(\dfrac{qh_1}{2}\right)+\dfrac{q^2}{4}\right] & \textit{as } h_2\to 0.\end{cases} \tag{2.6}$$

Remark 2.1. – The statement (a) guarantees the pointwise convergence of the spectrum of the discrete system towards the spectrum of the continuous one. Convergence (d) guarantees that the upper bound (c) (see also (1.14)) on the spectrum is sharp.

The statement (e) of the Proposition provides the pointwise limit of the spectrum when one of the mesh parameters tends to zero, the other one being fixed. Obviously, the eigenvalues

$\lambda^{p,q}(h_1, 0)$ correspond to the discretization of the continuous eigenvalue problem with respect to the variable x_1, i.e.,

$$\begin{cases} \varphi = \big(\varphi_0(x_2), \varphi_1(x_2), \dots, \varphi_J(x_2), \varphi_{J+1}(x_2)\big): \\ \quad -\left[\dfrac{\varphi_{j+1}(x_2) + \varphi_{j-1}(x_2) - 2\varphi_j(x_2)}{h_1^2}\right] - \dfrac{\partial^2 \varphi_j(x_2)}{\partial x_2^2} = \lambda \varphi_j(x_2), \\ \qquad 0 < x_2 < \pi, \ \ j = 1, \dots, J, \\ \varphi_j \equiv 0, \quad j = 0, J+1, \\ \varphi_j(x_2) = 0, \quad x_2 = 0, \pi, \ \ j = 0, \dots, J+1. \end{cases} \tag{2.7}$$

In a similar way, the eigenvalues $\lambda^{p,q}(0, h_2)$ correspond to the semi-discrete problem in which the Laplacian is discretized in the variable x_2 but not with respect to x_1.

When proving Theorems 1.1 and 1.2 the following identity from [8,9] will be useful.

Let us denote by ψ^ℓ the vector

$$\psi^\ell = \big(\psi_1^\ell, \dots, \psi_N^\ell\big), \quad \psi_j^\ell = \sin(j\ell h) \tag{2.8}$$

with $N + 1 = \pi/h$, for $\ell = 1, \dots, N$.

These are the eigenvectors of the 1-d discrete problem

$$\begin{cases} -\left[\dfrac{\psi_{j+1} + \psi_{j-1} - 2\psi_j}{h^2}\right] = \mu \psi_j, \quad j = 1, \dots, J, \\ \psi_0 = \psi_{J+1} = 0, \end{cases}$$

which is a finite-difference discretization of the 1-d continuous eigenvalue problem

$$\begin{cases} -\psi_{xx} = \mu \psi, \quad 0 < x < \pi, \\ \psi(0) = \psi(\pi) = 0. \end{cases}$$

The following identity holds:

LEMMA 2.1 [8,9]. – *For any $N \in \mathbb{N}$ and $h = \pi/(N+1)$ it follows that*

$$\frac{4}{h} \sin^2\left(\frac{h\ell}{2}\right) \sum_{j=1}^{N} \big|\psi_j^\ell\big|^2 = h \sum_{j=0}^{N} \left|\frac{\psi_{j+1}^\ell - \psi_j^\ell}{h}\right|^2 = \frac{\pi}{2(1 - \sin^2(h\ell/2))} \left|\frac{\psi_N^\ell}{h}\right|^2 \tag{2.9}$$

for all $\ell = 1, \dots, N$.

Remark 2.2. – Identity (2.9) provides the ratio between the total energy of the eigenvectors of the 1-d semi-discrete wave equation and the energy concentrated on the extreme $x = \pi$.

Indeed, the total energy of the eigenvectors is represented by the quantity

$$h \sum_{j=0}^{N} \left|\frac{\psi_{j+1}^\ell - \psi_j^\ell}{h}\right|^2$$

which is a discrete version of

$$\int_0^\pi \left|\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right|^2 \mathrm{d}x.$$

The energy concentrated on the extreme $x=\pi$ of the boundary is given by $|\psi_N^\ell/h|^2$ which is a discrete version of $|\,\mathrm{d}\psi(\pi)/\,\mathrm{d}x|^2$.

On the other hand, the first identity in (2.9) states that

$$h\sum_{j=0}^{J}\left|\frac{\psi_{j+1}^\ell-\psi_j^\ell}{h}\right|^2=\mu h\sum_{j=0}^{J}\left|\psi_j^\ell\right|^2$$

where $\mu=\mu^\ell(h)$ is the corresponding 1-d eigenvalue

$$\mu^\ell(h)=\frac{4}{h^2}\sin^2\left(\frac{h\ell}{2}\right).$$

Remark 2.3. – Note that the following holds as a consequence of (2.9):

$$h\sum_{j=1}^{N}\left|\psi_j^\ell\right|^2=\frac{\pi\sin^2(\ell Nh)}{2\sin^2(h\ell)}=\frac{\pi\sin^2(\ell(\pi-h))}{2\sin^2(h\ell)}=\frac{\pi}{2}. \tag{2.10}$$

Observe that the 2-d eigenvectors in (2.2) are products of vectors of the form (2.8). Thus identity (2.9) allows us to establish the corresponding 2-d observability identity.

PROPOSITION 2.2. – *Let* $\varphi^{p,q}(h_1,h_2)$ *be the eigenvector of* (1.13) *with* $(p,q)\in\{1,\ldots,J\}\times\{1,\ldots,K\}$ *and* $h_1,h_2>0$ *as in* (2.2). *Then*

$$\begin{aligned}&h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\frac{\varphi_{j+1,k}-\varphi_{j,k}}{h_1}\right|^2+\left|\frac{\varphi_{j,k+1}-\varphi_{j,k}}{h_2}\right|^2\right]\\&\quad=\frac{\pi}{2(1-\sin^2(ph_1/2))}h_2\sum_{k=0}^{K}\left|\frac{\varphi_{J,k}}{h_1}\right|^2+\frac{\pi}{2(1-\sin^2(qh_2/2))}h_1\sum_{j=0}^{J}\left|\frac{\varphi_{j,K}}{h_2}\right|^2.\end{aligned} \tag{2.11}$$

Remark 2.4. – In identity (2.11) we have avoided the superscripts (p,q) of φ to simplify the notation. The left hand side of (2.11) is the total energy of the eigenvector and it is a discrete version of the continuous energy

$$\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

The right hand side of (2.11) takes account of the energy concentrated on the observed subset of the boundary.

Proof of Proposition 2.2. – According to (2.2) we have $\varphi_{j,k}=\sin(jph_1)\sin(kqh_2)$. Then, in view of (2.9), we have

$$h_1h_2\sum_{j=0}^{J}\left|\frac{\varphi_{j+1,k}-\varphi_{j,k}}{h_1}\right|^2=h_2\frac{\pi}{2(1-\sin^2(ph_1/2))}\left|\frac{\varphi_{J,k}}{h_1}\right|^2$$

and

$$h_1 h_2 \sum_{k=0}^{K} \left| \frac{\varphi_{j,k+1} - \varphi_{j,k}}{h_2} \right|^2 = h_1 \frac{\pi}{2(1 - \sin^2(qh_2/2))} \left| \frac{\varphi_{j,K}}{h_2} \right|^2.$$

Therefore

$$h_1 h_2 \sum_{j=0}^{J} \sum_{k=0}^{K} \left[\left| \frac{\varphi_{j+1,k} - \varphi_{j,k}}{h_1} \right|^2 + \left| \frac{\varphi_{j,k+1} - \varphi_{j,k}}{h_2} \right|^2 \right]$$
$$= \frac{\pi}{2(1 - \sin^2(ph_1/2))} h_2 \sum_{k=0}^{K} \left| \frac{\varphi_{J,k}}{h_1} \right|^2 + \frac{\pi}{2(1 - \sin^2(qh_2/2))} h_1 \sum_{j=0}^{J} \left| \frac{\varphi_{j,K}}{h_2} \right|^2.$$

In view of identity (2.11), Theorems 1.1 and 1.2 are easy to prove. □

Proof of Theorem 1.1. – Given (p, q) and $h_1, h_2 > 0$ we consider the solution of (1.7) in separated variables associated to the eigenfunction $\varphi^{p,q}(h_1, h_2)$:

$$u = \cos\left(\sqrt{\lambda^{p,q}(h_1, h_2)}t\right)\varphi^{p,q}(h_1, h_2). \tag{2.12}$$

The initial energy $E_{h_1 h_2}(0)$ can be computed easily with the aid of identity (2.11):

$$E_{h_1,h_2}(0) = \frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \left[\left| \frac{\varphi_{j+1,k} - \varphi_{j,k}}{h_1} \right|^2 + \left| \frac{\varphi_{j,k+1} - \varphi_{j,k}}{h_2} \right|^2 \right]$$
$$= \frac{\pi}{4(1 - \sin^2(ph_1/2))} h_2 \sum_{k=0}^{K} \left| \frac{\varphi_{J,k}}{h_1} \right|^2 + \frac{\pi}{4(1 - \sin^2(qh_2/2))} h_1 \sum_{j=0}^{J} \left| \frac{\varphi_{j,K}}{h_2} \right|^2. \tag{2.13}$$

On the other hand, the energy concentrated on the boundary is given by

$$\int_0^T \cos^2\left(\sqrt{\lambda}t\right) \mathrm{d}t \left[h_1 \sum_{j=1}^{J} \left| \frac{\varphi_{j,K}}{h_2} \right|^2 + h_2 \sum_{k=1}^{K} \left| \frac{\varphi_{J,k}}{h_1} \right|^2 \right]. \tag{2.14}$$

Therefore, the ratio under consideration may be rewritten as follows:

$$Q(h_1, h_2) = E_{h_1,h_2}(0) \left(\int_0^T \left[h_1 \sum_{j=1}^{J} \left| \frac{u_{j,K}(t)}{h_2} \right|^2 + h_2 \sum_{k=1}^{K} \left| \frac{u_{J,k}(t)}{h_1} \right|^2 \right] \mathrm{d}t \right)^{-1}$$
$$= \frac{\frac{\pi}{4(1 - \sin^2(qh_2/2))} h_1 \sum_{j=0}^{J} \left| \frac{\varphi_{j,K}}{h_2} \right|^2 + \frac{\pi}{4(1 - \sin^2(ph_1/2))} h_2 \sum_{k=0}^{K} \left| \frac{\varphi_{J,k}}{h_1} \right|^2}{\int_0^T \cos^2\left(\sqrt{\lambda}t\right) \mathrm{d}t \left[h_1 \sum_{j=0}^{J} \left| \frac{\varphi_{j,k}}{h_2} \right|^2 + h_2 \sum_{k=0}^{K} \left| \frac{\varphi_{J,k}}{h_1} \right|^2 \right]}. \tag{2.15}$$

To prove Theorem 1.1 we take $p = J,\ q = K$ in the quotient (2.15), i.e., we consider the solution u of the form (2.12) associated to the largest eigenvalue. Let us now analyze the limit of the quotient $Q(h_1, h_2)$ as $h_1, h_2 \to 0$, i.e., when letting $p = J,\ q = K \to \infty$.

Taking into account that $\lambda^{p,q}(h_1,h_2)\to\infty$ it is easy to see that

$$\int_0^T \cos^2\left(\sqrt{\lambda}t\right)\mathrm{d}t \to \frac{T}{2}. \tag{2.16}$$

On the other hand

$$\lim_{h_2\to 0}\frac{\pi}{4(1-\sin^2(qh_2/2))}=\frac{\pi}{4}\lim_{h_2\to 0}\left[\frac{1}{1-\sin^2(\pi/2-h_2/2)}\right]=\infty.$$

In a similar way, we deduce that $\pi/[4(1-\sin^2(ph_1/2))]\to\infty$ as $h_1\to 0$.

In view of this, it is immediate to see that $Q(h_1,h_2)\to\infty$ as $h_1,h_2\to 0$. This concludes the proof of Theorem 1.1. □

Remark 2.5. – It is clear that the method of proof of Theorem 1.1 fails when p,q are restricted to satisfy

$$p\leqslant \delta J,\qquad q\leqslant \delta K$$

with $0<\delta<1$.

Indeed, in that case, the quotient in (2.15) can be easily bounded above by:

$$\frac{\max\left[\dfrac{\pi}{4\cos^2(\delta(\pi-h_2)/2)},\dfrac{\pi}{4\cos^2(\delta(\pi-h_1)/2)}\right]}{\int_0^T\cos^2\left(\sqrt{\lambda}t\right)\mathrm{d}t}\sim\frac{\pi}{4\cos^2(\delta\pi/2)(T/2-1/4\sqrt{\lambda})}$$

as $h_1,h_2\to 0$.

Note that the factor $\cos^2(\delta\pi/2)$ in the denominator tends to zero as $\delta\to 1$, and therefore the upper bound on the ratio tends to infinity. This explains the fact that Theorem 1.1 holds.

Proof of Theorem 1.2. – We choose $p=J$. Then we choose the value of q so that the eigenvalue $\lambda^{J,q}(h_1,h_2)$ is such that the solution u as in (2.12) is in the class $\mathcal{C}_\gamma(h_1,h_2)$. For that we need:

$$\begin{aligned}\lambda^{J,q}(h_1,h_2)&=4\left[\frac{1}{h_1^2}\sin^2\left(\frac{Jh_1}{2}\right)+\frac{1}{h_2^2}\sin^2\left(\frac{qh_2}{2}\right)\right]\\&=4\left[\frac{1}{h_1^2}\sin^2\left(\frac{\pi}{2}-\frac{h_1}{2}\right)+\frac{1}{h_2^2}\sin^2\left(\frac{qh_2}{2}\right)\right]\\&=4\left[\frac{\cos^2(h_1/2)}{h_1^2}+\frac{1}{h_2^2}\sin^2(qh_2/2)\right]\leqslant\gamma\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right),\end{aligned}\tag{2.17}$$

or, equivalently,

$$4\cos^2(h_1/2)-\gamma\leqslant\left[\gamma-4\sin^2(qh_2/2)\right]|h_1/h_2|^2. \tag{2.18}$$

Let us choose h_1,h_2 such that

$$\sup\left|\frac{h_2}{h_1}\right|<\sqrt{\frac{\gamma}{4-\gamma}}. \tag{2.19}$$

Of course this can be done by taking any $h_1\to 0$ and then $h_2=ah_1$ with $a<\sqrt{\gamma/(4-\gamma)}$.

Under assumption (2.19) it is clear that taking

$$q \leqslant \delta/h_2 \tag{2.20}$$

with $0<\delta<\pi$ small enough, (2.18) holds. This determines the choice of q.

With these choices of p and q let us now pass to the limit in the ratio $Q(h_1,h_2)$ in (2.15). It is easy to see that (2.16) holds. On the other hand, in view of (2.20),

$$\lim_{h_2\to 0}\frac{\pi}{4(1-\sin^2(qh_2/2))}=\lim_{h_2\to 0}\frac{\pi}{4\cos^2(qh_2/2)}<\infty, \tag{2.21}$$

while

$$\begin{aligned}\frac{\pi}{4(1-\sin^2(Jh_1/2))}&=\frac{\pi}{4\cos^2(Jh_1/2)}=\frac{\pi}{4\cos^2(\pi/2-h_1/2)}=\frac{\pi}{4\sin^2(h_1/2)}\\ &\sim\pi/h_1^2\to\infty\quad\text{as } h_1\to 0.\end{aligned} \tag{2.22}$$

In view of (2.21) and (2.22), to conclude that

$$Q(h_1,h_2)\to\infty\quad\text{as } h_1,h_2\to 0, \tag{2.23}$$

it is sufficient to show that

$$h_1^2\left[\left(h_1\sum_{j=0}^{J}\left|\frac{\varphi_{j,k}}{h_2}\right|^2\right)\left(h_2\sum_{k=0}^{K}\left|\frac{\varphi_{J,k}}{h_1}\right|^2\right)^{-1}\right]\to 0\quad\text{as } h_1,h_2\to 0. \tag{2.24}$$

In view of the form of the eigenvectors (2.2) and identity (2.9) it follows that

$$\begin{aligned}h_1\sum_{j=0}^{J}\left|\frac{\varphi_{j,k}}{h_2}\right|^2&=\frac{\sin^2(Kqh_2)}{h_2^2}h_1\sum_{j=0}^{J}\sin^2(jJh_1)\\ &=\frac{\pi\sin^2(Kqh_2)}{2h_2^2}=\frac{\pi\sin^2((\pi-h_2)q)}{2h_2^2}.\end{aligned} \tag{2.25}$$

On the other hand,

$$h_2\sum_{k=0}^{K}\left|\frac{\varphi_{J,k}}{h_1}\right|^2=\frac{\pi\sin^2(J^2h_1)}{2h_1^2}=\frac{\pi\sin^2(h_1)}{2h_1^2}. \tag{2.26}$$

Combining (2.25) and (2.26) we get

$$\begin{aligned}h_1^2\left[\left(h_1\sum_{j=0}^{J}\left|\frac{\varphi_{j,K}}{h_2}\right|^2\right)\left(h_2\sum_{k=0}^{K}\left|\frac{\varphi_{J,k}}{h_1}\right|^2\right)^{-1}\right]&=\frac{h_1^4\sin^2((\pi-h_2)q)}{h_2^2\sin^2(h_1)}\\ \sim\frac{h_1^2}{h_2^2}\sin^2\big((\pi-h_2)q\big)=\left|\frac{h_1}{h_2}\right|^2\sin^2(qh_2)&\to 0\quad\text{as } h_1,h_2\to 0\end{aligned} \tag{2.27}$$

provided

$$\sup\left|\frac{h_1}{h_2}\right|<\infty \tag{2.28}$$

and q is fixed independent of h_2.

Note that (2.19) and (2.28) are perfectly compatible. As we said above, it is sufficient to take $h_1 \to 0$ and $h_2 = ah_1$ with $a < \sqrt{\gamma/(4-\gamma)}$. $\square$

Remark 2.6. – Our proof works when $q = \mathrm{o}(1/h_2)$. More precisely, if instead of choosing q independent of h_2 as in the proof of Theorem 1.2, we choose q depending on h_2 such that $qh_2 \to 0$ as $h_2 \to 0$, the ratio Q still tends to infinity.

Remark 2.7. – Our proof of Theorem 1.2 works under the condition

$$\sup \left|\frac{h_2}{h_1}\right| < \sqrt{\frac{\gamma}{4-\gamma}} \tag{2.29}$$

or, the symmetric one,

$$\sup \left|\frac{h_1}{h_2}\right| < \sqrt{\frac{\gamma}{4-\gamma}}. \tag{2.30}$$

Condition (2.29) coincides with (2.19). By, symmetry, taking $q = K$ and $p = o(1/h_1)$, the proof of Theorem 1.2 works under assumption (2.30) as well.

Note that conditions (2.29) and (2.30) are sharp. Indeed, as indicated in Remark 2.4, to prove Theorem 1.2 for solutions generated by a single eigenvector we need to take $p = J$ (respectively $q = K$) since the ratio Q is uniformly bounded as soon as $p \leqslant \delta J$ and $q \leqslant \delta K$ with $0 < \delta < 1$. Then (2.29) (respectively (2.30)) is a necessary condition for the existence of eigenvalues in the range

$$\lambda \leqslant \gamma\big(h_1^{-2} + h_2^{-2}\big).$$

Observe that, if we take the same net spacing in x_1 and x_2, i.e., $h_1 = h_2 = h$, Theorem 1.2 only applies when $\gamma > 2$. As we shall see below, in this particular case $h_1 = h_2$, the observability inequality is uniform in the class $\mathcal{C}_\gamma(h_1, h_2)$ with $\gamma < 2$. Thus, the result of Theorem 1.2 is sharp.

3. Uniform observability estimates

This section is devoted to prove uniform observability estimates in classes of solutions in which the high frequencies have been filtered or truncated. Instead of applying directly 2-d discrete multiplier techniques we employ discrete Fourier series developments and 1-d discrete multipliers. First we prove some basic identities that are valid for all solutions of (1.7). Then we derive the uniform observability estimates by a suitable filtering of the high frequencies.

3.1. Preliminary identities

We develop solutions of (1.7) in Fourier series

$$u = \sum_{p=1}^{J}\sum_{q=1}^{K} \big(a_{p,q}e^{i\mu^{p,q}t} + b_{p,q}e^{-i\mu^{p,q}t}\big)\varphi^{p,q} \tag{3.1}$$

where

$$\mu^{p,q} = \sqrt{\lambda^{p,q}}. \tag{3.2}$$

In (3.1) we omit the dependence on h_1, h_2 of eigenvectors and eigenvalues to simplify the notation. When this will become necessary, we shall also use the subscript $\vec{h} = (h_1, h_2)$: $\varphi^{p,q} = \varphi_{\vec{h}}^{p,q}$, $\mu^{p,q} = \mu_{\vec{h}}^{p,q}, \ldots$

In view of the form of the eigenvectors (2.2) the solution u may be decomposed as

$$u = \sum_{p=1}^{J} \psi^p v^p \tag{3.3}$$

with

$$v^p = \sum_{q=1}^{K} \left(a_{p,q} e^{i\mu^{p,q}t} + b_{p,q} e^{-i\mu^{p,q}t}\right)\xi^q \tag{3.4}$$

and

$$\psi^p = \left(\psi_1^p, \ldots, \psi_J^p\right); \quad \psi_j^p = \sin(pjh_1), \tag{3.5}$$

$$\xi^q = \left(\xi_1^q, \ldots, \xi_K^q\right); \quad \xi_k^q = \sin(qkh_2). \tag{3.6}$$

The solution u of (1.7) can also be decomposed as

$$u = \sum_{q=1}^{K} \xi^q w^q, \tag{3.7}$$

with

$$w^q = \sum_{p=1}^{J} \left(a_{p,q} e^{i\mu^{p,q}t} + b_{p,q} e^{-i\mu^{p,q}t}\right)\psi^p. \tag{3.8}$$

Observe that for any $p = 1, \ldots, J$, $v^p = v$ solves the 1-d semi-discrete wave equation:

$$\begin{cases} v_k'' - \left[\dfrac{v_{k+1} + v_{k-1} - 2v_k}{h_2^2}\right] + \alpha^p v_k = 0, \quad 0 < t < T,\ k = 1, \ldots, K, \\ v_0 = v_{K+1} = 0, \quad 0 < t < T, \end{cases} \tag{3.9}$$

with

$$\alpha^p = \frac{4}{h_1^2} \sin^2\left(\frac{ph_1}{2}\right). \tag{3.10}$$

On the other hand, $w = w^q$ satisfies

$$\begin{cases} w_j'' - \left[\dfrac{w_{j+1} + w_{j-1} - 2w_j}{h_1^2}\right] + \beta^q w_j = 0, \quad 0 < t < T,\ j = 1, \ldots, J, \\ w_0 = w_{J+1} = 0, \quad 0 < t < T, \end{cases} \tag{3.11}$$

with

$$\beta^q = \frac{4}{h_2^2} \sin^2\left(\frac{qh_2}{2}\right). \tag{3.12}$$

Observe that the decompositions (3.3) and (3.7) are the discrete version of the classical decomposition of the solutions of the continuous 2-d wave equation in the square in solutions of a one-parameter family of 1-d wave equations.

The energy

$$F(t)=\frac{1}{2}\sum_{k=0}^{K}\left[\left|v_k'\right|^2+\left|\frac{v_{k+1}-v_k}{h_2}\right|^2+\alpha^p|v_k|^2\right] \tag{3.13}$$

is conserved for solutions of (3.9). More precisely,

$$F(t)=F(0),\quad \forall 0<t<T. \tag{3.14}$$

The conserved energy for solutions of (3.11) is given by

$$G(t)=\frac{1}{2}\sum_{j=0}^{J}\left[\left|w_j'\right|^2+\left|\frac{w_{j+1}-w_j}{h_1}\right|^2+\beta^q|w_j|^2\right], \tag{3.15}$$

i.e.,

$$G(t)=G(0),\quad \forall 0<t<T. \tag{3.16}$$

On the other hand, the energy conservation properties (3.14) and (3.16) for the 1-d systems (3.9) and (3.11) and the orthogonality properties of the 1-d eigenvectors:

$$\sum_{j=1}^{J}\psi_j^p\psi_j^{p'}=\sum_{j=0}^{J}\left(\psi_{j+1}^p-\psi_j^p\right)\left(\psi_{j+1}^{p'}-\psi_j^{p'}\right)=0 \tag{3.17}$$

for $p\neq p'$ and

$$\sum_{k=1}^{K}\xi_k^q\xi_k^{q'}=\sum_{k=0}^{K}\left(\xi_{k+1}^q-\xi_k^q\right)\left(\xi_{k+1}^{q'}-\xi_k^{q'}\right)=0 \tag{3.18}$$

for $q\neq q'$, imply the conservation property (1.9) for the energy E of solutions of the 2-d system (1.7).

The following identities hold:

LEMMA 3.1. – *For any solution v of* (3.9) *the following identity holds:*

$$\begin{aligned}&\frac{h_2}{2}\sum_{k=0}^{K}\int_0^T\left[\left|v_k'\right|^2+\left|\frac{v_{k+1}-v_k}{h_2}\right|^2-\alpha^p v_k v_{k+1}\right]\mathrm{d}t\\&\quad-\frac{h_2}{4}\sum\int_0^T\left|v_k'-v_{k+1}'\right|^2\mathrm{d}t+X_1(t)\Big|_0^T=\frac{\pi}{2}\int_0^T\left|\frac{v_K}{h_2}\right|^2\mathrm{d}t\end{aligned} \tag{3.19}$$

with

$$X_1(t)=h_2\sum_{k=0}^{K}k\left(\frac{v_{k+1}-v_{k-1}}{2}\right)v_k'. \tag{3.20}$$

In a similar way any solution w of (3.11) *satisfies*:

$$\frac{h_1}{2}\sum_{j=0}^{J}\int_0^T\left[\left|w_j'\right|^2+\left|\frac{w_{j+1}-w_j}{h_1}\right|^2-\beta^q w_j w_{j+1}\right]\mathrm{d}t$$

$$-\frac{h_1}{4}\sum_{j=0}^{J}\int_0^T\left|w_j'-w_{j+1}'\right|^2\mathrm{d}t+X_2(t)\Big|_0^T=\frac{\pi}{2}\int_0^T\left|\frac{w_J}{h_1}\right|^2\mathrm{d}t \tag{3.21}$$

with

$$X_2(t)=h_1\sum_{j=0}^{J}j\left(\frac{w_{j+1}-w_{j-1}}{2}\right)w_j'. \tag{3.22}$$

Proof. – We briefly sketch the proof of (3.19), since that of (3.21) is the same.

We proceed as in [8,9] using the discrete multiplier $k(v_{k+1}-v_{k-1})/2$ (which is the discrete version of the classical multiplier $x_2\partial v/\partial x_2$ for solutions of the continuous wave equation). Arguing as in [8,9] we obtain:

$$\frac{h_2}{2}\sum_{k=0}^{K}\int_0^T\left[v_k'v_{k+1}'+\left|\frac{v_{k+1}-v_k}{h_2}\right|^2+\alpha^p v_k k\left(\frac{v_{k+1}-v_{k-1}}{2}\right)\right]\mathrm{d}t+X_1(t)\Big|_0^T$$

$$=\frac{\pi}{2}\int_0^T\left|\frac{v_K}{h_2}\right|^2\mathrm{d}t. \tag{3.23}$$

We then observe that

$$\sum_{k=0}^{K}v_k k\left(\frac{v_{k+1}-v_{k-1}}{2}\right)=-\frac{1}{2}\sum_{k=0}^{K}v_k v_{k+1}. \tag{3.24}$$

On the other hand,

$$\sum_{k=0}^{K}v_k'v_{k+1}'=\sum_{k=0}^{K}\left|v_k'\right|^2-\frac{1}{2}\sum_{k=0}^{K}\left|v_k'-v_{k+1}'\right|^2. \tag{3.25}$$

Combining (3.23)–(3.25), identity (3.19) follows immediately. □

We may now establish the following identity for solutions of the 2-d system (1.7):

LEMMA 3.2. – *Every solution u of* (1.7) *satisfies*

$$\frac{h_1h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\left[2\left|u_{j,k}'\right|^2+\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2+\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2\right]\mathrm{d}t$$

$$-\frac{h_1h_2}{4}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\left|u_{j+1,k}'-u_{j,k}'\right|^2+\left|u_{j,k+1}'-u_{j,k}'\right|^2\right]\mathrm{d}t$$

$$-\frac{h_1h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\frac{(u_{j+1,k}-u_{j,k})}{h_1}\frac{(u_{j+1,k+1}-u_{j,k+1})}{h_1}\right.$$

$$+\frac{(u_{j,k+1}-u_{j,k})}{h_2}\frac{(u_{j+1,k+1}-u_{j+1,k})}{h_2}\bigg]\,\mathrm{d}t+X(t)\big|_0^T$$

$$(3.26)\qquad =\frac{\pi}{2}\left[h_2\sum_{k=0}^{K}\int_0^T\left|\frac{u_{J,k}}{h_1}\right|^2\mathrm{d}t+h_1\sum_{j=0}^{J}\int_0^T\left|\frac{u_{j,K}}{h_2}\right|^2\mathrm{d}t\right],$$

with

$$(3.27)\qquad X(t)=h_1h_2\sum_{j=1}^{J}\sum_{k=1}^{K}\left[k\left(\frac{u_{j,k+1}-u_{j,k-1}}{2}\right)u'_{j,k}+j\left(\frac{u_{j+1,k}-u_{j-1,k}}{2}\right)u'_{j,k}\right].$$

Proof. – Combining identity (3.19), the decomposition (3.3) and the orthogonality properties (3.17) we deduce that:

$$\frac{h_1h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\left|u'_{j,k}\right|^2+\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2-\frac{(u_{j+1,k}-u_{j,k})}{h_1}\frac{(u_{j+1,k+1}-u_{j,k+1})}{h_1}\right]\mathrm{d}t$$
$$-\frac{h_1h_2}{4}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|u'_{j,k}-u'_{j,k+1}\right|^2\mathrm{d}t+\frac{h_1h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}k(u_{j,k+1}-u_{j,k-1})u'_{j,k}\Big|_0^T$$
$$(3.28)\qquad =\frac{\pi h_1}{2}\sum_{j=0}^{J}\int_0^T\left|\frac{u_{j,K}}{h_2}\right|^2\mathrm{d}t.$$

In a similar way, one can show that:

$$\frac{h_1h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\left|u'_{j,k}\right|^2+\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2-\frac{(u_{j,k+1}-u_{j,k})}{h_2}\frac{(u_{j+1,k+1}-u_{j+1,k})}{h_2}\right]\mathrm{d}t$$
$$-\frac{h_1h_2}{4}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|u'_{j+1,k}-u'_{j,k}\right|^2\mathrm{d}t+\frac{h_1h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}j(u_{j+1,k}-u_{j-1,k})u'_{j,k}\Big|_0^T$$
$$(3.29)\qquad =\frac{\pi h_2}{2}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{J,k}}{h_1}\right|^2\mathrm{d}t.$$

Combining (3.28) and (3.29) we obtain (3.26).

Let us briefly check how (3.28) may be obtained. Let us for instance analyze the first term in the left hand side of (3.28). We have

$$\frac{h_1h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|u'_{j,k}\right|^2\mathrm{d}t=\frac{h_1h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|\sum_{p=1}^{J}\psi_j^p v_k^p\right|^2\mathrm{d}t$$
$$=\frac{h_1h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\sum_{p,p'=1}^{J}\psi_j^p\psi_j^{p'}v_k^p v_k^{p'}\,\mathrm{d}t.$$

Taking into account that, according to (2.10),

$$h_1 \sum_{j=0}^{J} \psi_j^p \psi_j^{p'} = \frac{\pi}{2} \delta_{p,p'} \tag{3.30}$$

we deduce that

$$\frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T |u'_{j,k}|^2 \,\mathrm{d}t = \frac{\pi}{2} h_2 \sum_{k=1}^{K} \sum_{p=1}^{K} \int_0^T \left| \frac{\mathrm{d}v_k^p}{\mathrm{d}t} \right|^2 \mathrm{d}t.$$

The same can be done with all the remainding terms in (3.28). The third term on the left hand side of (3.28) is slightly different. We have

$$\begin{aligned} &\frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left[\frac{(u_{j+1,k} - u_{j,k})}{h_1} \frac{(u_{j+1,k+1} - u_{j,k+1})}{h_1} \right] \mathrm{d}t \\ &\quad = \frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left[\frac{\sum_{p=1}^{J} [\psi_{j+1}^p - \psi_j^p] v_k^p}{h_1} \frac{\sum_{p=1}^{J} [\psi_{j+1}^p - \psi_j^p] v_{k+1}^p}{h_1} \right] \mathrm{d}t. \end{aligned}$$

We now take into account that, according to (3.17),

$$h_1 \sum_{j=0}^{J} \frac{(\psi_{j+1}^p - \psi_j^p)}{h_1} \frac{(\psi_{j+1}^{p'} - \psi_j^{p'})}{h_1} = 0 \quad \text{if } p \neq p'. \tag{3.31}$$

On the other hand,

$$h_1 \sum_{j=0}^{J} \left| \frac{\psi_{j+1}^p - \psi_j^p}{h_1} \right|^2 = \frac{4}{h_1} \sin^2\left(\frac{h_1 p}{2}\right) \sum_{j=1}^{J} |\psi_j^p|^2 = \alpha^p h_1 \sum_{j=1}^{J} |\psi_j^p|^2 = \frac{\pi \alpha^p}{2}. \tag{3.32}$$

Thus,

$$\frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left[\frac{(u_{j+1,k} - u_{j,k})}{h_1} \frac{(u_{j+1,k+1} - u_{j,k+1})}{h_1} \right] \mathrm{d}t = \sum_{p=1}^{J} \frac{\alpha^p \pi}{2} \int_0^T \left[\frac{h_2}{2} \sum_{k=0}^{K} v_k^p v_{k+1}^p \right] \mathrm{d}t.$$

This shows that (3.28) is simply a superposition of identities (3.19) for $p = 1, \ldots, J$. □

Remark 3.1. – Identity (3.26) is the semi-discrete version of the following well-known one for the solutions of the continuous wave equation (1.1) (see [14]):

$$\int_0^T \int_\Omega |u'|^2 \,\mathrm{d}x \,\mathrm{d}t + \int_\Omega u' x \cdot \nabla u \,\mathrm{d}x \Big|_0^T = \frac{1}{2} \int_0^T \int_{\partial\Omega} (x \cdot n) \left| \frac{\partial u}{\partial n} \right|^2 \mathrm{d}\Gamma \,\mathrm{d}t.$$

The first and last terms of (3.26) are reproduced in this continuous identity. The same can be said about $X(t)|_0^T$. Note however that (3.26) contains some extra terms that are due to the discretization.

We now need the *equipartition of energy identity* for the 2-d system (1.7):

LEMMA 3.3. – *Every solution u of* (1.7) *satisfies*

$$h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \left|u'_{j,k}\right|^2 dt$$

$$= Y(t)\big|_0^T + h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \left[\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2 + \left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2\right] dt \tag{3.33}$$

with

$$Y(t) = h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K} u_{j,k}u'_{j,k}. \tag{3.34}$$

Proof. – We multiply in (1.7) by $u_{j,k}$, add for $j = 1, \ldots, J,\ k = 1, \ldots, K$ and integrate with respect to $t \in (0, T)$. Identity (3.33) follows immediately taking into account that:

$$h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T u''_{j,k}u_{j,k}\, dt = h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K} u'_{j,k}u_{j,k}\big|_0^T - h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \left|u'_{j,k}\right|^2 dt,$$

$$h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \left(\frac{u_{j+1,k}+u_{j-1,k}-2u_{j,k}}{h_1^2}\right)u_{j,k}\, dt = -h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \left|\frac{u_{j,k}-u_{j+1,k}}{h_1}\right|^2 dt$$

and

$$h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \frac{(u_{j,k+1}+u_{j,k-1}-2u_{j,k})}{h_2^2}u_{j,k}\, dt = -h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2 dt.$$

□

Combining the identities in Lemmas 3.2 and 3.3 and the conservation of the discrete energy E (1.8) it follows that:

LEMMA 3.4. – *Every solution u of* (1.7) *satisfies*

$$TE(0) - \frac{h_1h_2}{4}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \left[\left|u'_{j,k+1}-u'_{j,k}\right|^2 + \left|u'_{j+1,k}-u'_{j,k}\right|^2\right] dt$$

$$+ \frac{h_1h_2}{2}\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \left[\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2 - \frac{(u_{j+1,k}-u_{j,k})(u_{j+1,k+1}-u_{j,k+1})}{h_1^2}\right.$$

$$\left. + \left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2 - \frac{(u_{j,k+1}-u_{j,k})(u_{j+1,k+1}-u_{j+1,k})}{h_2^2}\right] dt + Z(t)\big|_0^T$$

$$= \frac{\pi}{2}\left[h_1\sum_{j=0}^{J}\int_0^T \left|\frac{u_{j,K}}{h_2}\right|^2 dt + h_2\sum_{k=0}^{K}\int_0^T \left|\frac{u_{J,k}}{h_1}\right|^2 dt\right], \tag{3.35}$$

with

$$Z(t) = X(t) + \frac{Y(t)}{2}$$

$$= h_1 h_2 \sum_{j=0}^{J} \sum_{k=0}^{K} \left[k \frac{(u_{j,k+1} - u_{j,k})}{2} u'_{j,k} + j \frac{(u_{j+1,k} - u_{j,k})}{2} u'_{j,k} + \frac{1}{2} u_{j,k} u'_{j,k} \right]. \tag{3.36}$$

Remark 3.2. – Identity (3.35) is the semi-discrete version of the identity

$$T E(0) + \int_{\Omega} u_t \left(x \cdot \nabla u + \frac{u}{2} \right) \Big|_0^T = \frac{\pi}{2} \int_0^T \int_0^{\pi} \left[\left| \frac{\partial u}{\partial x_1}(\pi, x_2) \right|^2 + \left| \frac{\partial u}{\partial x_2}(x_1, \pi) \right|^2 \right] \tag{3.37}$$

that solutions of the continuous wave equation (1.1) satisfy. This identity may be proved using the multipliers $x \cdot \nabla u$ and u (see [12] and [14] for instance). In the case of the square $\Omega = (0, \pi) \times (0, \pi)$ it can also be obtained by means of Fourier decomposition and using 1-d multipliers. This is the method we have employed.

Note however that (3.35) contains two error terms:

$$\frac{h_1 h_2}{4} \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left[\left| u'_{j,k+1} - u'_{j,k} \right|^2 + \left| u'_{j+1,k} - u'_{j,k} \right|^2 \right] \mathrm{d}t$$

and

$$\frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left[\left| \frac{u_{j+1,k} - u_{j,k}}{h_1} \right|^2 - \frac{(u_{j+1,k} - u_{j,k})}{h_1} \frac{(u_{j+1,k+1} - u_{j,k+1})}{h_1} \right.$$
$$\left. + \left| \frac{u_{j,k+1} - u_{j,k}}{h_2} \right|^2 - \frac{(u_{j,k+1} - u_{j,k})}{h_2} \frac{(u_{j+1,k+1} - u_{j+1,k})}{h_2} \right] \mathrm{d}t.$$

Following the developments in [8,9] we shall get bounds on these error terms imposing upper bounds on the eigenvalues. Note however that upper bounds of the form $\lambda \leqslant \gamma(h_1^{-2} + h_2^{-2})$ will not be sufficient. We shall rather impose upper bounds of the form $\lambda \leqslant 2\gamma \min(h_1^{-2}, h_2^{-2})$, which allow to get upper bounds simultaneously on both the finite differences with respect to the x_1 and the x_2 directions.

3.2. Estimates on the error terms

As a consequence of Lemma 3.4 the following inequality holds:

LEMMA 3.5. – *Every solution u of* (1.7) *satisfies*

$$T E_{h_1,h_2}(0) - \frac{\Lambda}{4} \max\left(h_1^2, h_2^2\right) h_1 h_2 \sum_{j=0}^{J} \sum_{k=0}^{K} \int_0^T \left| u'_{j,k} \right|^2 \mathrm{d}t + Z(t) \Big|_0^T$$
$$\leqslant \frac{\pi}{2} \left[h_1 \sum_{j=0}^{J} \int_0^T \left| \frac{u_{j,k}}{h_2} \right|^2 \mathrm{d}t + h_2 \sum_{k=0}^{K} \int_0^T \left| \frac{u_{J,K}}{h_1} \right|^2 \right] \tag{3.38}$$

where Λ is the largest eigenvalue involved in the Fourier development of u.

Proof. – In view of Lemma 3.4, it is sufficient to estimate the remainders mentioned in Remark 3.1.

Let us consider first

$$R_1 = \sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \left[\left|u'_{j,k+1} - u'_{j,k}\right|^2 + \left|u'_{j+1,k} - u'_{j,k}\right|^2\right] \mathrm{d}t. \tag{3.39}$$

We claim that

$$R_1 \leqslant \Lambda \max\left(h_1^2, h_2^2\right) \sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \left|u'_{jk}\right|^2. \tag{3.40}$$

Indeed, in fact, the following more general fact is true:

LEMMA 3.6. – *Let I be a family of indexes (p,q). Let*

$$\Lambda = \max_{(p,q)\in I} \lambda^{p,q}. \tag{3.41}$$

Then

$$\begin{aligned}&\sum_{j=0}^{J}\sum_{k=0}^{K}\left[|\phi_{j,k+1} - \phi_{j,k}|^2 + |\phi_{j+1,k} - \phi_{j,k}|^2\right] \leqslant \Lambda \max\left(h_1^2, h_2^2\right) \sum_{j=0}^{J}\sum_{k=0}^{K} |\phi_{j,k}|^2,\\ &\forall \phi \in \operatorname*{span}_{(p,q)\in I}\left\{\varphi^{p,q}\right\}.\end{aligned} \tag{3.42}$$

Proof of Lemma 3.6. – We first observe that:

$$\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\frac{|\varphi_{j,k+1} - \varphi_{j,k}|^2}{h_2^2} + \frac{|\varphi_{j+1,k} - \varphi_{j,k}|^2}{h_1^2}\right] = \lambda \sum_{j=0}^{J}\sum_{k=0}^{K} |\varphi_{j,k}|^2 \tag{3.43}$$

when φ is an eigenvector of (1.13) with eigenvalue λ.

On the other hand, if φ and ψ are eigenvectors with non-equal indexes $(p,q) \neq (p',q')$ the following orthogonality properties hold:

$$\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\frac{(\varphi_{j,k+1} - \varphi_{j,k})}{h_2}\frac{(\psi_{j,k+1} - \psi_{j,k})}{h_2} + \frac{(\varphi_{j+1,k} - \varphi_{j,k})}{h_1}\frac{(\psi_{j+1,k} - \psi_{j,k})}{h_1}\right] = 0 \tag{3.44}$$

and

$$\sum_{j=0}^{J}\sum_{k=0}^{K} \varphi_{j,k}\psi_{j,k} = 0. \tag{3.45}$$

Combining (3.43)–(3.45) we deduce that

$$\begin{aligned}&\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\frac{\phi_{j,k+1} - \phi_{j,k}}{h_2}\right|^2 + \left|\frac{\phi_{j+1,k} - \phi_{j,k}}{h_1}\right|^2\right] \leqslant \Lambda \sum_{j=0}^{J}\sum_{k=0}^{K} |\phi_{j,k}|^2,\\ &\forall \phi \in \operatorname*{span}_{(p,q)\in I}\left\{\varphi^{p,q}\right\}.\end{aligned} \tag{3.46}$$

From (3.46), inequality (3.42) follows immediately taking into account that:

$$\sum_{j=0}^{J}\sum_{k=0}^{K}\left[|\phi_{j,k+1}-\phi_{j,k}|^2+|\phi_{j+1,k}-\phi_{j,k}|^2\right]$$
$$\leqslant \max\left(h_1^2,h_2^2\right)\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\frac{\phi_{j,k+1}-\phi_{j,k}}{h_2}\right|^2+\left|\frac{\phi_{j+1,k}-\phi_{j,k}}{h_1}\right|^2\right]. \qquad \square$$

We now return to the proof of Lemma 3.5.

In view of Lemma 3.6, estimate (3.40) is immediate. It is sufficient to apply (3.42) to $\phi=u(t)$ for any $t\in(0,T)$ and to integrate the resulting inequality for $t\in(0,T)$.

We now proceed to estimate

$$R_2=\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2-\frac{(u_{j+1,k}-u_{j,k})(u_{j+1,k+1}-u_{j,k+1})}{h_1^2}\right]\mathrm{d}t$$
$$+\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left[\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2-\frac{(u_{j,k+1}-u_{j,k})(u_{j+1,k+1}-u_{j+1,k})}{h_2^2}\right]\mathrm{d}t$$
$$=R_2^1+R_2^2. \tag{3.47}$$

Both terms have a similar structure. Let us analyze the first one R_2^1. We have

$$\left|\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\frac{(u_{j+1,k}-u_{j,k})(u_{j+1,k+1}-u_{j,k+1})}{h_1^2}\,\mathrm{d}t\right|$$
$$\leqslant\left(\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2\mathrm{d}t\right)^{1/2}\left(\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{j+1,k+1}-u_{j,k+1}}{h_1}\right|^2\mathrm{d}t\right)^{1/2}$$
$$=\sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2\mathrm{d}t. \tag{3.48}$$

In view of (3.48) we observe that $R_2^1\geqslant 0$. In a similar way we get $R_2^2\geqslant 0$. Therefore $R_2\geqslant 0$.

Combining identity (3.35) with (3.40) and the fact $R_2\geqslant 0$, inequality (3.38) follows immediately. $\square$

Combining Lemma 3.5 with the equipartition of energy identity the following holds:

LEMMA 3.7. – *Every solution of* (1.7) *satisfies*:

$$T\left(1-\frac{\Lambda}{4}\max\left(h_1^2,h_2^2\right)\right)E(0)+\widehat{Z}(t)\Big|_0^T$$
$$\leqslant\frac{\pi}{2}\left[\frac{h_1}{2}\sum_{j=0}^{J}\int_0^T\left|\frac{u_{j,K}}{h_2}\right|^2\mathrm{d}t+\frac{h_2}{2}\sum_{k=0}^{K}\int_0^T\left|\frac{u_{J,k}}{h_1}\right|^2\mathrm{d}t\right], \tag{3.49}$$

Λ being the largest eigenvalue entering in the Fourier development of u, and

$$\widehat{Z}(t) = Z(t) - \frac{\Lambda}{8}\max\bigl(h_1^2, h_2^2\bigr)Y(t). \tag{3.50}$$

Proof. – Combining the equipartition of energy identity and the conservation of energy property it follows that

$$h_1 h_2 \sum_{j=0}^{J}\sum_{k=0}^{K}\int_0^T \bigl|u'_{j,k}\bigr|^2\,\mathrm{d}t = TE(0) + \frac{1}{2}Y(t)\Big|_0^T. \tag{3.51}$$

Combining (3.38) and (3.51) inequality (3.49) follows immediately. □

We have to estimate now the quantity $\widehat{Z}$ in (3.49)–(3.50). The following holds:

LEMMA 3.8. – *Every solution u of* (1.7) *satisfies*

$$\begin{aligned}&\left(T\left(1 - \frac{\Lambda}{4}\max\bigl(h_1^2, h_2^2\bigr)\right) - 2\sqrt{2\pi^2 + (\eta^2 + 8|\eta|)/\lambda_1}\right)E(0)\\ &\quad\leqslant \frac{\pi}{2}\left[h_1 \sum_{j=0}^{J}\int_0^T \left|\frac{u_{j,K}}{h_2}\right|^2\mathrm{d}t + h_2 \sum_{k=0}^{K}\int_0^T\left|\frac{u_{J,k}}{h_1}\right|^2\mathrm{d}t\right],\end{aligned} \tag{3.52}$$

with λ_1 the least eigenvalue of (1.13), *Λ the largest eigenvalue entering in the Fourier expansion of u and*

$$\eta = \frac{1}{2} - \frac{\Lambda}{8}\max\bigl(h_1^2, h_2^2\bigr). \tag{3.53}$$

Proof. – Note that

$$\widehat{Z} = X + \eta Y \tag{3.54}$$

with X as in (3.27), Y as in (3.34) and η as in (3.53). In (3.54) and in the sequel we do not make explicit in the notation the dependence with respect to the time t of the functions under consideration.

We have

$$\widehat{Z} = h_1 h_2 \sum_{j=0}^{J}\sum_{k=0}^{K}\left[u'_{j,k}\left[\frac{k(u_{j,k+1} - u_{j,k-1})}{2} + \frac{j(u_{j+1,k} - u_{j-1,k})}{2} + \eta u_{j,k}\right]\right]. \tag{3.55}$$

Thus

$$\begin{aligned}|\widehat{Z}| &\leqslant h_1 h_2 \left(\sum_{j=0}^{J}\sum_{k=0}^{K}\bigl|u'_{j,k}\bigr|^2\right)^{1/2}\\ &\quad\times\left(\sum_{j=0}^{J}\sum_{k=0}^{K}\left|\frac{k(u_{j,k+1} - u_{j,k-1})}{2} + \frac{j(u_{j+1,k} - u_{j-1,k})}{2} + \eta u_{j,k}\right|^2\right)^{1/2}.\end{aligned} \tag{3.56}$$

On the other hand

$$
\begin{aligned}
&\sum_{j=0}^{J}\sum_{k=0}^{K}\left|\frac{k(u_{j,k+1}-u_{j,k-1})}{2}+\frac{j(u_{j+1,k}-u_{j-1,k})}{2}+\eta u_{j,k}\right|^2\\
&\quad\leqslant \sum_{j=0}^{J}\sum_{k=0}^{K}\Bigg[\left|\frac{k(u_{j,k+1}-u_{j,k-1})}{2}+\frac{j(u_{j+1,k}-u_{j-1,k})}{2}\right|^2+\eta^2|u_{j,k}|^2\\
&\qquad+\eta k(u_{j,k+1}-u_{j,k-1})u_{j,k}+\eta j(u_{j+1,k}-u_{j-1,k})u_{j,k}\Bigg]\\
&\quad\leqslant \sum_{j=0}^{J}\sum_{k=0}^{K}\Bigg[2\pi^2\Bigg[\left|\frac{u_{j,k+1}-u_{j,k-1}}{2h_2}\right|^2+\left|\frac{u_{j+1,k}-u_{j-1,k}}{2h_1}\right|^2\Bigg]\\
&\qquad+\eta^2|u_{j,k}|^2-\eta(u_{j,k+1}u_{j,k}+u_{j+1,k}u_{j,k})\Bigg]\\
&\quad\leqslant \sum_{j=0}^{J}\sum_{k=0}^{K}\Bigg[2\pi^2\Bigg(\frac{1}{2}\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2+\frac{1}{2}\left|\frac{u_{j,k}-u_{j,k-1}}{h_2}\right|^2+\frac{1}{2}\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2\\
&\qquad+\frac{1}{2}\left|\frac{u_{j,k}-u_{j-1,k}}{h_1}\right|^2\Bigg)+\eta^2|u_{j,k}|^2-\eta(u_{j,k+1}u_{j,k}+u_{j+1,k}u_{j,k})\Bigg]\\
&\quad=2\pi^2\sum_{j=0}^{J}\sum_{k=0}^{K}\Bigg[\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2+\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2\Bigg]+\big(\eta^2+8|\eta|\big)\sum_{j=0}^{J}\sum_{k=0}^{K}|u_{j,k}|^2
\end{aligned}
$$

$$
\leqslant\left[2\pi^2+\frac{(\eta^2+8|\eta|)}{\lambda_1}\right]\sum_{j=0}^{J}\sum_{k=0}^{K}\Bigg[\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2+\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2\Bigg] \tag{3.57}
$$

where λ_1 is the least eigenvalue of (1.13).

Combining (3.56) and (3.57) we deduce that

$$
\big|\widehat{Z}\big|\leqslant\sqrt{2\pi^2+\frac{(\eta^2+8|\eta|)}{\lambda_1}}\,E(0). \tag{3.58}
$$

In view of (3.58) we deduce that

$$
\big|\widehat{Z}(t)\big|_0^T\big|\leqslant\big|\widehat{Z}(0)\big|+\big|\widehat{Z}(T)\big|\leqslant 2\sqrt{2\pi^2+(\eta^2+8|\eta|)/\lambda_1}\,E(0). \tag{3.59}
$$

Combining (3.49) and (3.59) we deduce (3.52). □

3.3. Uniform boundary observability

In view of Lemma 3.7 it is easy to obtain uniform (as $h_1, h_2 \to 0$) observability inequalities. For, we introduce the following classes of solutions of (1.7) for any $0<\beta<1$:

$$
\widehat{\mathcal{C}}_\beta(h_1,h_2)=\left\{\begin{array}{l} u \text{ solution of (1.7) generated by the eigenvectors of (1.13)}\\ \text{such that } \lambda\max\big(h_1^2,h_2^2\big)\leqslant 4\beta \end{array}\right\}. \tag{3.60}
$$

The following holds:

THEOREM 3.1. – *Let* $0<\beta<1$. *Assume that*

$$T>\frac{2\sqrt{2\pi^2+c(\beta)}}{1-\beta}=T(\beta) \tag{3.61}$$

with

$$c(\beta)=\left[\frac{1}{4}(1-\beta)^2+4(1-\beta)\right]\Big/\lambda_1. \tag{3.62}$$

Then, there exists $C=C(\beta,T)>0$ *such that*

$$E_{h,h_2}(0)\leqslant C(\beta,T)\int_0^T\left[h_1\sum_{j=0}^{J}\left|\frac{u_{j,K}(t)}{h_2}\right|^2+h_2\sum_{k=0}^{K}\left|\frac{u_{J,k}(t)}{h_1}\right|^2\right]\mathrm{d}t \tag{3.63}$$

holds for every solution u *of* (1.7) *in the class* $\widehat{\mathcal{C}}_\beta(h_1,h_2)>0$ *and every* $h_1,h_2>0$.

Moreover, the constant $C(\beta,T)$ *may be taken to be*

$$C(\beta,T)=\frac{\pi}{2[T(1-\beta)-2\sqrt{2\pi^2+c(\beta)}]}. \tag{3.64}$$

Proof. – According to inequality (3.52) and taking into account that

$$\frac{\Lambda}{4}\max\left(h_1^2,h_2^2\right)=\beta$$

in the class $\widehat{\mathcal{C}}_\beta(h_1,h_2)$, it follows that

$$\begin{aligned}&\left(T(1-\beta)-2\sqrt{2\pi^2+(\eta^2+8|\eta|)/\lambda_1}\right)E_{h_1,h_2}(0)\\&\quad\leqslant\frac{\pi}{2}\int_0^T\left[h_1\sum_{j=0}^{J}\left|\frac{u_{j,K}}{h_2}\right|^2+h_2\sum_{k=0}^{K}\left|\frac{u_{J,k}}{h_1}\right|^2\right]\mathrm{d}t\end{aligned} \tag{3.65}$$

with

$$\eta=\frac{1}{2}-\frac{\beta}{2}=\frac{1}{2}(1-\beta).$$

The statement of Theorem 3.1 follows immediately from (3.65). □

Remark 3.3. – In the definition (3.62) of $c(\beta)$, the least eigenvalue λ_1 depends on h_1,h_2. However, as $h_1,h_2\to 0$, λ_1 converges to the least eigenvalue for $-\Delta$ in $H_0^1((0,\pi)\times(0,\pi))$. Thus,

$$\lambda_1\to 2\quad\text{as } h_1,h_2\to 0.$$

Thus the minimal observability time remains bounded as $h_1,h_2\to 0$. Moreover

$$T(\beta)\to\frac{2\sqrt{2\pi^2+c^*(\beta)}}{1-\beta}\quad\text{as } h_1,h_2\to 0$$

with

$$c^*(\beta)=\frac{1}{8}(1-\beta)^2+2(1-\beta).$$

Remark 3.4. – The minimal observability time $T(\beta)$ satisfies

$$T(\beta) \to \infty \quad \text{as } \beta \to 1. \tag{3.66}$$

Moreover

$$T(\beta) \to 2\sqrt{2\pi^2 + c(0)} = 2\sqrt{2\pi^2 + 17/4\lambda_1} \quad \text{as } \beta \to 0. \tag{3.67}$$

This indicates that:

(a) We loose the observability inequality as $\beta \to 1$ not only because individual eigenvectors are less and less observable but also because we need more time to get upper bounds on the interaction between different Fourier modes. This is in agreement with the 1-d results of [8,9].

(b) As β decreases the observability time decreases. This is also in agreement with the 1-d results. However, the estimate $T(\beta)$ on the observability time is not sharp since we do not recover the observability time $2\sqrt{2}\pi$ needed for the continuous wave equation as $\beta \to 0$ and $h_1, h_2 \to 0$.

At this respect note that, in the 1-d case, the sharp observability time (twice the lenght of the interval) was recovered.

This lack of optimality is due to the estimates of the proof of Lemma 3.9 on $\widehat{Z}$ and more precisely to the terms

$$h_1, h_2 \sum_{j=0}^{J} \sum_{k=0}^{K} \left[\eta^2 |u_{j,k}|^2 - \eta(u_{j,k+1} u_{j,k} + u_{j+1,k} u_{j,k})\right]. \tag{3.68}$$

Note that in the context of the continuous wave equation the corresponding term is

$$\left(\eta^2 - \eta\right) \int_{\Omega} |u|^2 \,\mathrm{d}x_1 \,\mathrm{d}x_2$$

which is non-positive (and therefore may be neglected) as soon as $\eta^2 - \eta \leqslant 0$. In Theorem 3.1, $\eta = \frac{1}{2}(1-\beta)$. Thus

$$\eta^2 - \eta = \frac{1}{4}(1-\beta)^2 - \frac{1}{2}(1-\beta) = \frac{1}{2}(1-\beta)\left[\frac{1}{2}(1-\beta) - 1\right]$$
$$= -\frac{1}{4}(1-\beta)^2 \leqslant 0 \quad \text{for } 0 \leqslant \beta \leqslant 1.$$

This sign property does not seem to hold for the discrete quantity (3.68). Thus, we are obliged to get upper bounds on the absolute value of this quantity.

In Section 3.4 we shall see how a compactness-uniqueness argument may be used to improve the observability time, so that, as $\beta \to 0$, we recover the observability time $2\sqrt{2}\pi$ for the continuous wave equation.

Remark 3.5. – Note that the uniform observability inequality of Theorem 3.1 holds in the subspaces of solutions of the form $\widehat{\mathcal{C}}_\beta(h_1, h_2)$ and not in the subsapces $\mathcal{C}_\gamma(h_1, h_2)$ introduced in (1.16). We recall that, according to Theorem 1.1, the uniform observability inequality may not hold in $\mathcal{C}_\gamma(h_1, h_2)$ whatever γ is, if we do not impose further conditions on h_1, h_2.

3.4. Optimality of the uniform observability inequality

As we have seen in Remark 3.3, the estimate provided by Theorem 3.1 on the observability time is suboptimal.

Let us now analyze the optimality of Theorem 3.1 in what concerns the frequencies involved in the class $\widehat{\mathcal{C}}_\beta(h_1,h_2)$. For, we compare Theorem 3.1 to the counterexamples of Theorems 1.1 and 1.2.

In to order to analyze the optimality of Theorem 3.1 we distinguish the following three cases:

Case 1: $h_1=h_2=h$;

Case 2: $h_2=\ell h_1$, with $\ell>1$;

Case 3: $h_2=\ell h_1$, with $\ell<1$.

Case 1: When $h_1=h_2=h$, it is easy to check that

$$\widehat{\mathcal{C}}_\beta(h_1,h_2)=\mathcal{C}_{2\beta}(h_1,h_2). \tag{3.69}$$

Therefore, Theorem 3.1 guarantees the observability in the classes $\mathcal{C}_\gamma\,(h_1,h_2)$ for any $\gamma<2$.

According to Remark 1.2, the result is sharp since the uniform observability fails for any $\gamma>2$. The case $\gamma=2$ corresponding to $\beta=1$ remains open.

Case 2: When $h_2=\ell h_1$ with $\ell>1$, the condition

$$\lambda\max\bigl(h_1^2,h_2^2\bigr)\leqslant 4\beta \tag{3.70}$$

that characterizes the class $\widehat{\mathcal{C}}_\beta(h_1,h_2)$ can be rewritten as

$$\lambda\leqslant 4\beta h_2^{-2}=\frac{4\beta}{h_2^2}. \tag{3.71}$$

On the other hand, the condition characterizing the class $\mathcal{C}_\gamma(h_1,h_2)$ is

$$\lambda\leqslant\gamma\left[\frac{1}{h_1^2}+\frac{1}{h_2^2}\right]=\frac{\gamma}{h_2^2}\bigl(1+\ell^2\bigr). \tag{3.72}$$

We have $\mathcal{C}_\gamma(h_1,h_2)\subset\widehat{\mathcal{C}}_\beta(h_1,h_2)$ as soon as $\gamma(1+\ell^2)\leqslant 4\beta$ or, in other words,

$$\gamma\leqslant\frac{4\beta}{\ell^2+1}. \tag{3.73}$$

In view of Theorem 3.1 we deduce that, under the condition $h_2=\ell h_1$ with $\ell>1$, the uniform observability holds in $\mathcal{C}_\gamma(h_1,h_2)$ as soon as

$$\gamma<\frac{4}{1+\ell^2}\Leftrightarrow\ell<\sqrt{\frac{4-\gamma}{\gamma}}. \tag{3.74}$$

On the other hand, as we have seen in Remark 1.2, the counterexample of Theorem 1.2 applies as soon as

$$\sup\left|\frac{h_1}{h_2}\right|=\frac{1}{\ell}<\sqrt{\frac{\gamma}{4-\gamma}}.$$

Thus, the result of Theorem 3.1 is sharp. The limit case $\beta = 1$ which corresponds to $\gamma = 4/(1+\ell^2)$ remains open.

Case 3: By symmetry, the situation is the same as in Case 2 above. Thus Theorem 3.1 is sharp and the limit case $\beta = 1$ remains open.

Summarizing, assuming that $h_2 = \ell h_1$, with $\ell \geqslant 1$ the following results hold in the classes $\mathcal{C}_\gamma(h_1, h_2)$:

(a) If $\gamma > 4/(1+\ell^2)$, the uniform observability fails in any time interval;

(b) If $\gamma < 4/(1+\ell^2)$, the uniform observability holds for $T > T(\gamma)$, with $T(\gamma) > 0$ large enough;

(c) The limit case $\gamma = 4/(1+\ell^2)$ is open.

4. Uniform observability: Sharp observability time

4.1. Motivation and main result

As we indicated in Remark 3.3, the minimal observability time obtained in Theorem 3.1 is suboptimal since, even in the limit case when $\beta = 0$, we do not get the optimal controllability time $2\sqrt{2}\pi$ for the continuous wave equation. Let us recall that the controllability time we get in the class $\widehat{\mathcal{C}}_\beta(h_1, h_2)$, with $0 < \beta < 1$, is given by the expression

$$T(\beta) = \frac{2\sqrt{2\pi^2 + c(\beta)}}{1-\beta} \tag{4.1}$$

with

$$c(\beta) = \left[\frac{1}{4}(1-\beta)^2 + 4(1-\beta)\right] \Big/ \lambda_1. \tag{4.2}$$

Recall however (see Remark 3.4.b) that the term $c(\beta)$ in this expression is due to the upper bounds we get for the quantity in (3.67). Note that this expression is of the order of the discrete L^2-norm of the solutions. Therefore, it is a lower order term when compared to the energy. This suggests that a compactness-uniqueness argument may allow us to get rid of it. This compactness-uniqueness principle guarantees that, if an observability inequality holds up to a compact additive remainder, then the remainder may be removed provided a suitable uniqueness or unique-continuation result holds. This has been applied systematically in the context of PDE (see for instance Appendix I in [14]). We apply it here in the discrete framework.

The following result holds:

THEOREM 4.1. – *Let* $0 < \beta < 1$. *Assume that*

$$T > T^*(\beta) = \frac{2\sqrt{2}\pi}{1-\beta}. \tag{4.3}$$

Then, there exists $C^* = C^*(\beta, T) > 0$ *such that*

$$E_{h_1,h_2}(0) \leqslant C^*(\beta, T) \int_0^T \left[h_1 \sum_{j=0}^{J} \left| \frac{u_{j,K}(t)}{h_2} \right|^2 + h \sum_{k=0}^{K} \left| \frac{u_{J,k}(t)}{h_1} \right|^2 \right] \mathrm{d}t \tag{4.4}$$

holds for every solution u *of* (1.7) *in the class* $\widehat{\mathcal{C}}_\beta(h_1, h_2)$ *and* $h_1, h_2 > 0$.

Remark 4.1. – Note that $T^*(\beta) < T(\beta)$. Moreover, $T(\beta) \to 2\sqrt{2}\pi$ as $\beta \to 0$. This indicates that, when working in the class of solutions of (1.7) in which only the eigenvectors corresponding to

$$\lambda \max\bigl(h_1^2, h_2^2\bigr) = \mathrm{o}(1)$$

enter, then we recover the uniform observability inequality for any $T > 2\sqrt{2}\pi$.

Remark 4.2. – As indicated above, the proof of (4.4) uses a compactness-uniqueness argument. Thus the constant $C^*(\beta, T)$ in (4.4) is not explicit anymore.

The rest of this section is devoted to prove Theorem 4.1. But before doing that we need to analyze carefully how solutions of the semi-discretized system (1.7) converge to the solutions of the continuous system (1.1) as $h_1, h_2 \to 0$. This is the object of the following section.

4.2. Convergence of the solutions of the semi-discrete systems

Consider a family $u = u(\vec{h}, t)$ of solutions of (1.7) depending on the parameter $\vec{h} = (h_1, h_2) \to 0$. Solutions $u(\vec{h}, t)$ may be developed in Fourier series as follows

$$u\bigl(\vec{h}, t\bigr) = \sum_{p=1}^{J}\sum_{q=1}^{K} u^{p,q}\bigl(\vec{h}; t\bigr)\varphi_{\vec{h}}^{p,q}, \tag{4.5}$$

where $\varphi_{\vec{h}}^{p,q}$ are the eigenvectors of (1.13) and $u^{p,q}(\vec{h}; t)$ the time-dependent Fourier coefficients.

In the following Proposition we describe how a uniformly bounded family of solutions of (1.7), weakly converges as $h_1, h_2 \to 0$ to a solution of finite energy of the continuous wave equation (1.1).

PROPOSITION 4.1. – *Let* $u = u(\vec{h}, t)$ *be a family of solutions of* (1.7) *depending on the parameters* $\vec{h} = (h_1, h_2) \to (0, 0)$, *whose energies are uniformly bounded, i.e.,*

$$E_{h_1,h_2}(0) \leqslant C, \quad \forall \vec{h} = (h_1, h_2). \tag{4.6}$$

Then, by extracting a suitable subsequence $(h_1, h_2) \to (0, 0)$ *we may guarantee that*

$$u^{p,q}\bigl(\vec{h}, \cdot\bigr) \rightharpoonup u^{p,q} \quad \textit{weakly in } H^1(0, T),\ \textit{as } \vec{h} \to (0, 0),\ \forall (p, q); \tag{4.7}$$

$$u = \sum_{p,q \geqslant 1} u^{p,q}(t) \sin(px_1) \sin(qx_2) \ \textit{solves } (1.1); \tag{4.8}$$

$$u \in C\bigl([0, T]; H_0^1(\Omega)\bigr) \cap C^1\bigl([0, T]; L^2(\Omega)\bigr); \tag{4.9}$$

$$E(0) \leqslant \liminf_{\vec{h} \to (0,0)} E_{h_1,h_2}(0); \tag{4.10}$$

$$h_1 h_2 \sum_{j=1}^{J}\sum_{k=1}^{K} \bigl|u_{j,k}\bigl(\vec{h}, t\bigr)\bigr|^2 \to \bigl\|u(t)\bigr\|_{L^2(\Omega)}^2 \quad \textit{in } L^\infty(0, T). \tag{4.11}$$

Remark 4.3. – Note that in Proposition 4.1 we do not impose to the solutions to belong to any class $\widehat{\mathcal{C}}_\beta(h_1,h_2)$ or $\mathcal{C}_\gamma(h_1,h_2)$ in which the high frequencies have been filtered. Thus, Proposition 4.1 applies to any bounded family of solutions of (1.7).

Observe that in (4.11) we state the uniform (in time) convergence of the L^2-norms of the discrete solutions to the L^2-norm of the continuous one. Note that the left hand side in (4.11) is written in terms of the values of the solutions $u(\vec{h},t)$ at the mesh points although, it could also be written in terms of its Fourier coefficients.

Proof of Proposition 4.1. – We shall view the two-parameter family of Fourier coefficients $\{u^{p,q}\}$ as functions defined for $t\in[0,T]$ and $\vec{h}=(h_1,h_2)$ and with values in the Hilbert space ℓ^2 of square summable sequences:

$$\ell^2=\left\{\left\{a^{p,q}\right\}\colon \sum_{p,q\geqslant 1}\left|a^{p,q}\right|^2<\infty\right\},$$

endowed with the canonical norm. Note however that for any $\vec{h}=(h_1,h_2)$ fixed, $u^{p,q}$ is only well-defined for (p,q) with $1\leqslant p\leqslant J$ and $1\leqslant q\leqslant K$. Thus, we set $u^{p,q}(\vec{h},\cdot)=0$ for all $\vec{h}$ and (p,q) such that $p\geqslant J+1$ or $q\geqslant K+1$.

In view of the uniform bound (4.6) and the conservation of the energies we deduce that:

$$\left\{u^{p,q}\left(\vec{h},\cdot\right)\right\} \text{ is uniformly bounded in } W^{1,\infty}\left(0,T;\ell^2\right) \text{ as } \vec{h}\to 0. \tag{4.12}$$

To simplify the notation a bit more we shall denote by $\vec{u}(\vec{h};t)$ the sequence of Fourier coefficients $\{u^{p,q}(\vec{h},\cdot)\}_{p,q\geqslant 1}$. By extracting subsequences we obtain

$$\vec{u}\left(\vec{h};t\right)\rightharpoonup\vec{u}(t)=\left\{u^{p,q}(t)\right\}_{p,q\geqslant 1}\quad\text{weakly in } H^1\left(0,T;\ell^2\right),\ \text{as } \vec{h}\to 0. \tag{4.13}$$

Clearly the function (4.8) is then a solution of the wave equation (1.1).

Note also that, according to (4.6) and the conservation of energy we also have:

$$\begin{aligned}&h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\frac{u_{j+1,k}-u_{j,k}}{h_1}\right|^2+\left|\frac{u_{j,k+1}-u_{j,k}}{h_2}\right|^2\right](t)\leqslant C,\\&\forall 0\leqslant t\leqslant T,\ \forall\vec{h}=(h_1,h_2).\end{aligned} \tag{4.14}$$

We now observe that the eigenvectors $\vec{\varphi}^{p,q}(\vec{h})$ satisfy

$$\begin{aligned}h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\left[\left|\frac{\varphi^{p,q}_{j+1,k}-\varphi^{p,q}_{j,k}}{h_1}\right|^2+\left|\frac{\varphi^{p,q}_{j,k+1}-\varphi^{p,q}_{j,k}}{h_2}\right|^2\right]&=\lambda^{p,q}h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\left|\varphi^{p,q}_{j,k}\right|^2\\&=\frac{\lambda^{p,q}\pi^2}{4}.\end{aligned} \tag{4.15}$$

On the other hand, the eigenvectors are orthogonal in the corresponding scalar products. Thus, (4.14) is equivalent to

$$\sum_{p,q}\lambda^{p,q}\left(\vec{h}\right)\left|u^{p,q}\left(\vec{h},t\right)\right|^2\leqslant C,\quad\forall 0\leqslant t\leqslant T,\ \forall\vec{h}. \tag{4.16}$$

Observe that (4.16) provides a uniform bound for $\vec{u}(\vec{h})$ in a weighted ℓ^2-norm. However, the weights are actually the eigenvalues $\lambda^{p,q}(\vec{h})$ and therefore, they depend on $\vec{h}$.

In order to avoid the dependence of the weights on the parameter $\vec{h}$ we observe that the following lower bound holds for the eigenvalues: There exists $c > 0$ such that

$$c\left(p^2+q^2\right) \leqslant \lambda^{p,q}\left(\vec{h}\right), \quad \forall 1 \leqslant p \leqslant J,\ 1 \leqslant q \leqslant K,\ \forall \vec{h}. \tag{4.17}$$

To prove (4.17) we observe that $\lambda_{\vec{h}}^{p,q}$ can be written in the following form

$$\lambda_{\vec{h}}^{p,q} = p^2\left[\frac{\sin(ph_1/2)}{ph_1/2}\right]^2 + q^2\left[\frac{\sin(qh_2/2)}{qh_2/2}\right]^2.$$

On the other hand $0 \leqslant ph_1/2,\ qh_2/2 \leqslant \pi/2$ for any (p,q) and (h_1,h_2). Thus, using the fact that

$$\sin(z)/z \geqslant c > 0, \quad \forall z \in [0,\pi/2],$$

we conclude that (4.17) holds.

Combining (4.16)–(4.17) we deduce that

$$\sum_{p,q\geqslant 0}\left(p^2+q^2\right)\left|u^{p,q}\left(\vec{h};t\right)\right| \leqslant C, \quad \forall 0 \leqslant t \leqslant T,\ \forall \vec{h}. \tag{4.18}$$

Let us now introduce the following Hilbert space of sequences:

$$\hbar^1 = \left\{\left\{a^{p,q}\right\} \in \ell^2\colon \sum_{p\geqslant 0}\sum_{q\geqslant 0}\left(p^2+q^2\right)\left|a^{p,q}\right|^2 < \infty\right\}, \tag{4.19}$$

endowed with the canonical norm. It is easy to see that $\hbar^1$ is compactly imbedded in ℓ^2. On the other hand, according to (4.18),

$$\left\{\vec{u}\left(\vec{h},\cdot\right)\right\}_{\vec{h}} \text{ is uniformly bounded in } C\left([0,T];\hbar^1\right). \tag{4.20}$$

Using the classical Aubin–Lions compactness result (see for instance J. Simon [16]) we deduce that

$$\vec{u}\left(\vec{h},\cdot\right) \text{ is relatively compact in } C\left([0,T];\ell^2\right). \tag{4.21}$$

Thus, we deduce that

$$\vec{u}\left(\vec{h},\cdot\right) \rightharpoonup \vec{u}(\cdot) \quad \text{weakly in } H^1\left(0,T;\ell^2\right) \cap L^2\left(0,T;\hbar^1\right) \tag{4.22}$$

and

$$\vec{u}\left(\vec{h},\cdot\right) \to \vec{u}(\cdot) \quad \text{strongly in } C\left([0,T];\ell^2\right). \tag{4.23}$$

We now observe that

$$\frac{\pi^2}{4}\sum_{p,q}\left|u^{p,q}\left(\vec{h},t\right)\right|^2 = h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}\left|u_{j,k}\left(\vec{h};t\right)\right|^2. \tag{4.24}$$

Indeed, this is due to the fact that

$$h_1 h_2 \sum_{j,k} |\varphi_{j,k}|^2 = \pi^2/4$$

for the eigenvectors (2.2) of system (1.13) and the fact that $\varphi^{p,q}$ is orthogonal to $\varphi^{p',q'}$ in the discrete L^2-norm when $(p,q) \neq (p',q')$. In view of identity (4.24) the convergence (4.11) follows immediately from (4.23), since $u^{p,q}(t)$ are precisely the Fourier coefficients of u in the basis $\{\sin(px_1)\sin(qx_2)\}$ of $L^2(\Omega)$.

Observe that, according to the bounds (4.12) and (4.20) the subsequence may be extracted so that

$$\begin{cases} \vec{u}(\vec{h};0) \rightharpoonup \vec{u}_0 & \text{weakly in } \hbar^1, \\ \vec{u}'(\vec{h};0) \rightharpoonup \vec{u}_1 & \text{weakly in } \ell^2, \end{cases} \tag{4.25}$$

for some $(\vec{u}_0, \vec{u}_1) \in \hbar^1 \times \ell^2$. Note that, in view of (4.23), we have

$$\vec{u}(0) = \vec{u}_0. \tag{4.26}$$

We now use the fact that

$$\left(u^{p,q}\right)'' + \lambda^{p,q} u^{p,q} = 0$$

which combined with (4.20) and the fact that

$$\lambda^{p,q}\left(\vec{h}\right) \leqslant C\left(p^2+q^2\right), \quad \forall \vec{h},\ \forall (p,q) \tag{4.27}$$

for a suitable $C > 0$, implies that

$$\vec{u}''\left(\vec{h};\cdot\right) \text{ is uniformly bounded in } C\left([0,T];\hbar^{-1}\right); \tag{4.28}$$

where $\hbar^{-1}$ is the following Hilbert space of sequences:

$$\hbar^{-1} = \left\{ \left\{a^{p,q}\right\} \in \ell^2 \colon \sum_{p \geqslant 1} \sum_{q \geqslant 1} \left(p^2+q^2\right)^{-1} \left|a^{p,q}\right|^2 < \infty \right\}. \tag{4.29}$$

Combining (4.12), (4.28) and the compactness results mentioned above we deduce that

$$\vec{u}'\left(\vec{h},\cdot\right) \to \vec{u}'(\cdot) \quad \text{strongly in } C\left([0,T];\hbar^{-1}\right). \tag{4.30}$$

Combining (4.25) and (4.30) we deduce that

$$\vec{u}'(0) = \vec{u}_1. \tag{4.31}$$

The fact that $(\vec{u}_0, \vec{u}_1) \in \hbar^1 \times \ell^2$ is equivalent to saying that the limit initial data are of finite energy. Consequently the limit solution of the continuous wave equation (1.1) is in the class (4.9).

To conclude the proof of Proposition 4.1 it is sufficient to prove the lower semicontinuity of the energy property (4.10).

In view of (4.24) we have

$$E_{h_1,h_2}(0)=\frac{\pi^2}{8}\sum_{p,q}\left[\lambda^{p,q}(\vec{h})\left|u^{p,q}(\vec{h},0)\right|^2+\left|\left(u^{p,q}\right)'(\vec{h},0)\right|^2\right]. \tag{4.32}$$

In view of the second convergence in (4.25) we deduce that

$$\sum_{p,q}\left|\left(u^{p,q}\right)'(0)\right|^2\leqslant\liminf_{\vec{h}\to(0,0)}\sum_{p,q}\left|\left(u^{p,q}\right)'(\vec{h},0)\right|^2. \tag{4.33}$$

On the other hand, taking into account that

$$\lambda^{p,q}(\vec{h})\to p^2+q^2\quad\text{as }\vec{h}\to(0,0)$$

and

$$u^{p,q}(\vec{h},0)\to u^{p,q}(0)\quad\text{as }\vec{h}\to(0,0),$$

for all (p,q), we deduce that

$$\sqrt{\lambda^{p,q}(\vec{h})}\,u^{p,q}(\vec{h},0)\to\sqrt{(p^2+q^2)}\,u^{p,q}(0)\quad\text{as }\vec{h}\to(0,0).$$

Therefore

$$\sum_{p,q\geqslant1}\left(p^2+q^2\right)\left|u^{p,q}(0)\right|^2\leqslant\liminf_{(h_1,h_2)\to(0,0)}\sum_{p,q\geqslant1}\lambda^{p,q}(\vec{h})\left|u^{p,q}(\vec{h},0)\right|^2. \tag{4.34}$$

Obviously

$$E(0)=\frac{\pi^2}{8}\sum_{p,q\geqslant1}\left[\left|\left(u^{p,q}\right)'(0)\right|^2+\left(p^2+q^2\right)\left|u^{p,q}(0)\right|^2\right].$$

Therefore, combining (4.33) and (4.34) we deduce that (4.10) holds. □

In order to apply the compactness-uniqueness argument to prove Theorem 4.1 we need also to analyze the behavior of the normal derivatives of the solutions of the semi-discrete systems (1.7) as $\vec{h}\to(0,0)$. This is done in the following proposition:

PROPOSITION 4.2. – *Let* $u=u(\vec{h},t)$ *be a family of solutions of* (1.7) *depending on* $\vec{h}=(h_1,h_2)\to(0,0)$ *satisfying* (4.6). *Let* u *be any of the solutions of* (1.1) *obtained as limits when* $\vec{h}\to(0,0)$ *as in the statement of Proposition* 4.1. *Then*

$$\int_{\Gamma_0\times(0,T)}\left|\frac{\partial u}{\partial n}\right|^2\mathrm{d}\sigma\,\mathrm{d}t\leqslant\liminf_{h_1,h_2\to0}\int_0^T\left[\frac{h_1}{2}\sum_{j=0}^{J}\left|\frac{u_{j,K}}{h_2}\right|^2+\frac{h_2}{2}\sum_{j=0}^{K}\left|\frac{u_{J,k}}{h_1}\right|^2\right]\mathrm{d}t, \tag{4.35}$$

where u *is the limit solution of the wave equation* (1.1) *given by Proposition* 4.1.

Remark 4.4. – In the statement of this Proposition we do not make explicit the dependence of the solutions on the parameters h_1, h_2 to make the notation simpler.

Proof. – Along this proof we do not make explicit the dependence of solutions in the parameter $\vec{h}$. We decompose the integral on the left hand side of (4.35) as follows

$$\int_0^T \left[\frac{h_1}{2} \sum_{j=0}^{J} \left| \frac{u_{j,k}}{h_2} \right|^2 + \frac{h_2}{2} \sum_{k=0}^{K} \left| \frac{u_{J,k}}{h_1} \right|^2 \right] = \frac{1}{2}\big(I_1 + I_2\big) \tag{4.36}$$

with

$$I_1 = \int_0^T \left[h_1 \sum_{j=0}^{J} \left| \frac{u_{j,K}}{h_2} \right|^2 \right] \mathrm{d}t; \qquad I_2 = \int_0^T \left[h_2 \sum_{k=0}^{K} \left| \frac{u_{J,k}}{h_1} \right|^2 \right] \mathrm{d}t.$$

In view of the decompositions (3.3)–(3.7) we may write

$$u = \sum_{p,q} u^{p,q}(t)\psi^p \xi^q$$

and therefore

$$u_{j,k} = \sum_{p,q} u^{p,q}(t)\psi_j^p \xi_k^q.$$

Thus

$$\begin{aligned} I_1 &= h_1 \int_0^T \sum_{j=0}^{J} \left| \left(\sum_{p,q} u^{p,q}(t)\psi_j^p \xi_K^q \right)^2 h_2^{-2} \right| \mathrm{d}t = h_1 \int_0^T \sum_{j=0}^{J} \left| \left(\sum_q \left(\sum_p u^{p,q}(t)\psi_j^p \right) \xi_K^q \right)^2 h_2^{-2} \right| \mathrm{d}t \\ &= h_1 \int_0^T \sum_{j=0}^{J} \left[\sum_q \left(\sum_p u^{p,q}(t)\psi_j^p \right) \xi_K^q \right] \left[\sum_{q'} \left(\sum_{p'} u^{p',q'}(t)\psi_j^{p'} \right) \xi_K^{q'} \right] h_2^{-2}\, \mathrm{d}t \\ &= \frac{h_1}{h_2^2} \int_0^T \sum_{j=0}^{J} \left[\sum_{q,q'} \xi_K^q \xi_K^{q'} \sum_{p,p'} u^{p,q}(t) u^{p',q'}(t) \psi_j^p \psi_j^{p'} \right] \mathrm{d}t. \end{aligned}$$

In view of (3.30) we deduce that

$$\begin{aligned} I_1 &= \frac{\pi}{2h_2^2} \int_0^T \left[\sum_{q,q'} \xi_K^q \xi_K^{q'} \sum_p u^{p,q}(t) u^{p,q'}(t) \right] \mathrm{d}t \\ &= \frac{\pi}{2} \sum_p \int_0^T \left(\sum_q u^{p,q}(t) \xi_K^q \right)^2 h_2^{-2}\, \mathrm{d}t. \end{aligned} \tag{4.37}$$

Recall that

$$\begin{aligned} \xi_K^q / h_2 &= \sin\big(qKh_2\big)/h_2 = \sin\big(q(\pi - h_2)\big)/h_2 = -\cos(q\pi)\sin(qh_2)/h_2 \\ &\to -q\cos(q\pi) \quad \text{as } h_2 \to 0, \end{aligned} \tag{4.38}$$

for all $q \geqslant 0$.

We claim that, for any $p \geqslant 1$,

$$\sum_q \frac{u^{p,q}(t)\xi_K^q}{h_2} \rightharpoonup -\sum_q q\cos(q\pi)u^{p,q}(t) \quad \text{weakly in } L^2(0,T),\ \text{as } \vec{h} \to (0,0). \tag{4.39}$$

In the right hand side of (4.40) $u^{p,q}$ denote the coefficients of the limit solution of the wave equation provided by Proposition 4.1.

Assuming for the moment that (4.39) holds, by lower semicontinuity of the $L^2(0,T)$-norm we deduce that

$$\int_0^T \left| \sum_q q\cos(q\pi)u^{p,q}(t) \right|^2 \mathrm{d}t \leqslant \liminf_{\vec{h}\to(0,0)} \int_0^T \left| \sum_q \frac{u^{p,q}(t)\xi_K^q}{h_2} \right|^2 \mathrm{d}t \tag{4.40}$$

for all $p \geqslant 1$, and therefore

$$\begin{aligned} \frac{\pi}{2} \sum_p \int_0^T \left| \sum_q q\cos(q\pi)u^{p,q}(t) \right|^2 \mathrm{d}t &\leqslant \frac{\pi}{2} \liminf_{\vec{h}\to(0,0)} \sum_p \int_0^T \left| \sum_q \frac{u^{p,q}(t)\xi_k^q}{h_2} \right|^2 \mathrm{d}t \\ &= \liminf_{\vec{h}\to(0,0)} I_1. \end{aligned} \tag{4.41}$$

Note however that:

$$\frac{\pi}{2} \sum_p \int_0^T \left| \sum_q q\cos(q\pi)u^{p,q}(t) \right|^2 \mathrm{d}t = \int_0^T \int_{(0,\pi)\times\{y=\pi\}} \left| \frac{\partial u}{\partial n} \right|^2 \mathrm{d}\sigma\, \mathrm{d}t, \tag{4.42}$$

u being the limit solution of the continuous wave equation (1.1).

Combining (4.41) and (4.42) we have

$$\int_0^T \int_{(0,\pi)\times\{y=\pi\}} \left| \frac{\partial u}{\partial n} \right|^2 \mathrm{d}\sigma\, \mathrm{d}t \leqslant \liminf_{\vec{h}\to(0,0)} I_1. \tag{4.43}$$

In a similar way one gets

$$\int_0^T \int_{\{x=\pi\}\times(0,\pi)} \left| \frac{\partial u}{\partial n} \right|^2 \mathrm{d}\sigma\, \mathrm{d}t \leqslant \liminf_{\vec{h}\to(0,0)} I_2. \tag{4.44}$$

Let us now prove the statement (4.39).

In view of assumption (4.35), for any $p \geqslant 1$ fixed,

$$\sum_q u^{p,q}(\vec{h},t) \frac{\xi_K^q(\vec{h})}{h_2} \quad \text{is bounded in } L^2(0,T). \tag{4.45}$$

Let us introduce the notation

$$v(\vec{h},t) = \sum_q u^{p,q}(\vec{h},t)\xi^q(\vec{h}). \tag{4.46}$$

Note that v depends on p. However, since p is fixed in this discussion, we do not make explicit this dependence in the notation.

It is easy to see that v solves the 1-d semi-discrete wave equation:

$$\begin{cases} v_k'' - \left[\dfrac{v_{k+1} + v_{k-1} - 2v_k}{h_2^2}\right] + \alpha^p v_k = 0, & 0 < t < T,\ k = 1, \dots, K, \\ v_0 = v_{K+1} = 0, & 0 < t < T, \end{cases} \tag{4.47}$$

with

$$\alpha^p = \frac{4}{h_1^2} \sin^2\left(\frac{ph_1}{2}\right). \tag{4.48}$$

The energy of solutions of system (4.47) is given by

$$F_{h_2}(t) = \frac{1}{2}\left[h_2 \sum_{k=0}^{K} \left|v_k'\right|^2 + \left|\frac{v_{k+1} - v_k}{h_2}\right|^2 + \alpha^p |v_k|^2\right]. \tag{4.49}$$

It is easy to check that F_{h_2} remains constant in time for any $h_2 > 0$.

In view of the uniform boundedness condition (4.6) we have

$$F_{h_2}(t) \leqslant C, \quad \forall t \in (0, T),\ \forall h_2 > 0. \tag{4.50}$$

As a consequence of Proposition 4.1, the weak limit as $h_2 \to 0$ of $v\,(h_2, t)$ is given by

$$v\big(x_2, t\big) = \sum_{q \geqslant 1} u^{p,q}(t) \sin\big(q x_2\big) \tag{4.51}$$

and solves the 1-d wave equation

$$\begin{cases} v'' - \dfrac{\partial^2 v}{\partial x_2^2} + p^2 v = 0, & 0 < t < T,\ 0 < x_2 < \pi, \\ v(0, t) = v(\pi, t) = 0, & 0 < t < T. \end{cases} \tag{4.52}$$

The statement (4.39) is equivalent to showing that

$$\frac{v_K(h_2, t)}{h_2} \rightharpoonup \frac{\partial v}{\partial x_2}(\pi, t) \quad \text{weakly in } L^2(0, T). \tag{4.53}$$

In view of the uniform bound (4.45), it is sufficient to show that

$$\frac{v_K(h_2, t)}{h_2} \to \frac{\partial v}{\partial x_2}(\pi, t) \quad \text{in } \mathcal{D}'(0, T). \tag{4.54}$$

The semi-discrete and continuous wave equations (4.47) and (4.52) can be written respectively as follows:

$$\begin{cases} -\left[\dfrac{v_{k+1} + v_{k-1} - 2v_k}{h_2^2}\right] = -v_k'' - \alpha^p v_k, & 0 < t < T,\ k = 1, \dots, K, \\ v_0 = v_{K+1} = 0, & 0 < t < T, \end{cases} \tag{4.55}$$

and

$$\begin{cases} -\dfrac{\partial^2 v}{\partial x_2^2} = -v'' - p^2 v, & 0 < t < T,\ 0 < x_2 < \pi, \\ v(0,t) = v(\pi,t) = 0, & 0 < t < T. \end{cases} \tag{4.56}$$

As a consequence of Proposition 4.1 we have that the right hand side of (4.55) weakly converges to the right hand side of (4.56) in $H^{-1}(0,T;\ell^2)$. More precisely

$$\begin{gathered} \big(u^{p,q}\big)''\big(\vec{h},t\big) + \alpha^p u^{p,q}\big(\vec{h},t\big) \rightharpoonup \big(u^{p,q}\big)''(t) + p^2 u^{p,q}(t) \\ \text{weakly in } H^{-1}(0,T;\ell^2) \text{ as } \vec{h} \to (0,0). \end{gathered} \tag{4.57}$$

Therefore the problem is reduced to the following one which is of elliptic nature.

Consider $f \in L^2(0,\pi)$ and the elliptic problem

$$\begin{cases} -w_{xx} = f, & 0 < x < \pi, \\ w(0) = w(\pi) = 0, \end{cases} \tag{4.58}$$

with

$$f = \sum_{q \geqslant 1} f^q \sin(qx). \tag{4.59}$$

On the other hand let us consider the discretized problems

$$\begin{cases} -\left[\dfrac{w_{k+1} + w_{k-1} - 2w_k}{h^2}\right] = f_k(h), & k = 1,\dots,K, \\ w_0 = w_{K+1} = 0, \end{cases} \tag{4.60}$$

with $h > 0$ such that $(K+1) = \pi/h \in \mathbb{Z}$, and with

$$f_k = \sum_{q \geqslant 1} f^q(h) \sin(qkh), \tag{4.61}$$

where

$$\big\{f^q(h)\big\}_{q\geqslant 1} \rightharpoonup \big\{f^q\big\}_{q \geqslant 1} \quad \text{weakly in } \ell^2 \text{ as } h \to 0. \tag{4.62}$$

To conclude (4.56) and to complete the proof of Proposition 4.2 it is sufficient to show that the following holds:

LEMMA 4.1. – *Under the assumptions above the solution $w(h)$ of* (4.60) *is such that*

$$-\frac{w_K(h)}{h} \to w_x(\pi) \quad \textit{as } h \to 0 \tag{4.63}$$

where w is solution of (4.58).

This lemma states the convergence of the discrete normal derivatives of the solutions of the discrete problem (4.60) towards the normal derivative of the solution of (4.58).

As an immediate consequence of (4.57) and of this Lemma we deduce that (4.54) holds.

Proof of Lemma 4.1. – The solution $w_k(h)$ of (4.60) can be written explicitly:

$$w_k(h) = \sum_{q=1}^{K} \frac{f^q(h)}{\lambda^q(h)} \sin(qkh). \tag{4.64}$$

On the other hand, the solution w of (4.58) is as follows:

$$w(x) = \sum_{q \geqslant 1} \frac{f^q}{q^2} \sin(qx). \tag{4.65}$$

Thus, the statement (4.63) is equivalent to

$$-\sum_{q=1}^{K} \frac{f^q(h)}{\lambda^q(h)} \frac{\sin(qkh)}{h} \to \sum_{q \geqslant 1} \frac{f^q}{q} \cos(q\pi) \quad \text{as } h \to 0. \tag{4.66}$$

Note that

$$\sin(qkh) = \sin(q\pi - qh) = -\sin(qh)\cos(q\pi).$$

Thus, (4.66) is equivalent to

$$\sum_{q \geqslant 1} \left[\frac{f^q(h)}{\lambda^q(h)} \frac{\sin(qh)}{h} - \frac{f^q}{q} \right] \cos(q\pi) \to 0. \tag{4.67}$$

In (4.67) and in the sequel we assume that $f^q(h) = 0$ for $q \geqslant k+1$.

In view of (4.62) it is easy to see that, for any $M > 0$ fixed

$$\sum_{q=1}^{M} \left[\frac{f^q(h)}{\lambda^q(h)} \frac{\sin(qh)}{h} - \frac{f^q}{q} \right] \cos(q\pi) \to 0.$$

Thus, it is sufficient to check that, for any $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that

$$\left| \sum_{q \geqslant M(\varepsilon)+1} \left[\frac{f^q(h)}{\lambda^q(h)} \frac{\sin(qh)}{h} - \frac{f^q}{q} \right] \cos(q\pi) \right| \leqslant \varepsilon \quad \text{uniformly as } h \to 0. \tag{4.68}$$

Taking into account that $\{f^q\} \in \ell^2$ it is easy to see that

$$\left| \sum_{q \geqslant M(\varepsilon)+1} f^q \frac{\cos(q\pi)}{q} \right| \leqslant \varepsilon/2$$

for $M(\varepsilon) > 0$ large enough.

Let us now analyze the remainder:

$$\begin{aligned} \left| \sum_{q \geqslant M(\varepsilon)+1} \frac{f^q(h)}{\lambda^q(h)} \frac{\sin(qh)}{h} \cos(q\pi) \right| &\leqslant \sum_{q \geqslant M(\varepsilon)+1} \frac{|f^q(h)|}{\lambda^q(h)} q \\ &\leqslant \left(\sum_{q \geqslant M(\varepsilon)+1} \left| f^q(h) \right|^2 \right)^{1/2} \left(\sum_{q \geqslant M(\varepsilon)+1} \left| \frac{q}{\lambda^q(h)} \right|^2 \right)^{1/2}. \end{aligned} \tag{4.69}$$

In view of (4.62) and (4.69) we observe that it is sufficient to show that

$$\sum_{q \geqslant M(\varepsilon)+1} \left| \frac{q}{\lambda^q(h)} \right|^2 \to 0 \quad \text{as } M(\varepsilon) \to \infty \text{ uniformly on } h. \tag{4.70}$$

It is easy to see that (4.70) holds taking into account that $\lambda^q(h) \geqslant cq^2$ for some $c > 0$ for all $h > 0$ and $q = 1, \ldots, k$.

This completes the proof of Lemma 4.1. □

The proof of Proposition 4.2 is now complete. □

PROPOSITION 4.3. – *Under the assumptions of Proposition* 4.2, *if moreover*:

$$\int_0^T \left[\frac{h_1}{2} \sum_{j=0}^{J} \left| \frac{u_{j,K}}{h_2} \right|^2 + \frac{h_2}{2} \sum_{j=0}^{K} \left| \frac{u_{J,k}}{h_1} \right|^2 \right] \mathrm{d}t \to 0 \quad \textit{as } \vec{h} \to 0$$

and $T > 2\pi$, *then, necessarily, the limit solution* u *of the wave equation* (1.1) *is identically zero, i.e.,* $u \equiv 0$.

Proof. – According to Proposition 4.2 it follows that the limit u satisfies, in addition to the wave equation (1.1):

$$\partial u / \partial n = 0 \quad \text{on } \Gamma_0 \times (0, T).$$

In view of the fact that $T > 2\pi$, applying Holmgren's uniqueness Theorem we deduce that $u \equiv 0$. □

Remark 4.5. – The statement (4.39) we have proved along the proof of Proposition 4.2 is worth underlying. It may be rigorously stated as follows. Let us consider the 1-d wave equation

$$\begin{cases} v'' - v_{xx} + \alpha v = 0, & 0 < t < T,\ 0 < x < \pi, \\ v(0,t) = v(\pi,t) = 0, & 0 < t < T, \end{cases} \tag{4.71}$$

and its space semi-discretization

$$\begin{cases} v_k'' - \left[\dfrac{v_{k+1} + v_{k-1} - 2v_k}{h^2} \right] + \alpha v_k = 0, & 0 < t < T,\ k = 1, \ldots, K, \\ v_0 = v_{K+1} = 0, & 0 < t < T, \end{cases} \tag{4.72}$$

with $\alpha \geqslant 0$. Let $\{\vec{v}(h,t)\}$ be the Fourier coefficients of the solutions of the semi-discrete system (4.72), depending on the mesh-size $h = \pi/(K+1) \to 0$. Assume that the discrete energy of solutions of solutions of (4.72) is uniformly bounded. Then, by extracting subsequences, we have

$$\{\vec{v}(h,t)\} \to \{\vec{v}(t)\} \quad \text{as } h \to 0,\ \text{weakly in } L^2(0, T; \hbar^1),$$

$\vec{v}$ being the Fourier coefficients of a solution v of finite-energy of (4.71) and

$$-v_K(h,t)/h \to v_x(\pi, t) \quad \text{as } h \to 0,\ \text{weakly in } L^2(0,T).$$

4.3. Proof of the main result

This section is devoted to prove Theorem 4.1. Let us recall that, as we have seen in Lemma 3.8 and, more precisely, in identity (3.49), we have

$$T(1-\beta)E_{h_1,h_2}(0)+\widehat{Z}(t)\Big|_0^T \leqslant \frac{\pi}{2}\int_0^T \left[\frac{h_1}{2}\sum_{j=0}^{J}\left|\frac{u_{j,K}}{h_2}\right|^2+\frac{h_2}{2}\sum_{k=0}^{K}\left|\frac{u_{J,k}}{h_1}\right|^2\right]\mathrm{d}t \tag{4.73}$$

for all $0<\beta<1$ and all solution $u\in\widehat{\mathcal{C}}_\beta(h_1,h_2)$.

On the other hand, in view of (3.53)–(3.54),

$$\widehat{Z}=Z+\eta Y \tag{4.74}$$

with Y and Z as in (3.34) and (3.36) respectively.

In Lemma 3.9 we obtained upper bounds on $\widehat{Z}$. We proceed now in a slightly different way, considering the terms of the form

$$h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}|u_{j,k}|^2$$

as lower order terms. Going back to (3.56)–(3.57) we deduce that, for any $\varepsilon>0$ there exists a constant $C=C(\varepsilon,\beta)>0$ such that

$$\big|\widehat{Z}(t)\big|\leqslant\big(\sqrt{2}\pi+\varepsilon\big)E_{h_1,h_2}(0)+Ch_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}|u_{j,k}|^2(t) \tag{4.75}$$

for every $0\leqslant t\leqslant T$ and every solution u in the class $\widehat{\mathcal{C}}_\beta(h_1,h_2)$.

Combining (4.73) and (4.75) we deduce that

$$\begin{aligned}&\big(T(1-\beta)-2\big(\sqrt{2}\pi+\varepsilon\big)\big)E_{h_1,h_2}(0)\\ &\leqslant\frac{\pi}{2}\int_0^T\left[\frac{h_1}{2}\sum_{j=0}^{J}\left|\frac{u_{j,K}}{h_2}\right|^2+\frac{h_2}{2}\sum_{k=0}^{K}\left|\frac{u_{J,k}}{h_1}\right|^2\right]\mathrm{d}t+C\max_{0\leqslant t\leqslant T}\left\{h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}|u_{j,k}|^2(t)\right\}.\end{aligned} \tag{4.76}$$

Assume that

$$T>\frac{2(\sqrt{2}\pi+\varepsilon)}{1-\beta}. \tag{4.77}$$

Obviously, this is always the case for some $\varepsilon>0$ sufficiently small provided $T>T^*(\beta)$ as in (4.3).

Then, in view of (4.76) it is sufficient to show the existence of a constant $C>0$, independent of $h_1,h_2>0$, such that

$$\left\|h_1h_2\sum_{j=0}^{J}\sum_{k=0}^{K}|u_{j,k}|^2\right\|^2_{L^\infty(0,T)}\leqslant C\int_0^T\left[h_1\sum_{j=0}^{J}\left|\frac{u_{j,K}}{h_2}\right|^2+h_2\sum_{k=0}^{K}\left|\frac{u_{J,k}}{h_1}\right|^2\right]\mathrm{d}t, \tag{4.78}$$

for every $u\in\widehat{\mathcal{C}}_\beta(h_1,h_2)$ and all $h_1,h_2>0$.

We argue by contradiction. If (4.78) does not hold, there is a consequence $(h_1, h_2) \to (0, 0)$ and a sequence of solutions of the corresponding problem (1.7), that we denote by u (without making explicit its dependence on h_1, h_2, to simplify the notation) such that:

$$\int_0^T \left[h_1 \sum_{j=0}^{J} \left| \frac{u_{j,K}}{h_2} \right|^2 + h_2 \sum_{k=0}^{K} \left| \frac{u_{J,k}}{h_1} \right|^2 \right] \mathrm{d}t \to 0, \quad \text{as } h_1, h_2 \to 0; \tag{4.79}$$

$$\left\| h_1 h_1 \sum_{j=0}^{J} \sum_{k=0}^{K} |u_{j,k}|^2 \right\|_{L^\infty(0,T)} = 1. \tag{4.80}$$

In view of (4.76)–(4.80) we deduce that $E_{h_1,h_2}(0)$ is uniformly bounded. More precisely,

$$E_{h_1,h_2}(0) \leqslant C, \quad \forall h_1, h_2 > 0. \tag{4.81}$$

In view of (4.80)–(4.81) and applying Proposition 4.1 we deduce the existence of a solution u of the continuous wave equation (1.1) such that

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} = 1. \tag{4.82}$$

However, according to (4.79) and Proposition 4.3 we also have $u \equiv 0$. These two facts are in contradiction. This completes the proof of the existence of the constant C independent of $\vec{h}$ satisfying (4.78) and therefore the proof of Theorem 4.1 as well.

REFERENCES

[1] M. ASCH and G. LEBEAU, Geometrical aspects of exact boundary controllability for the wave equation. A numerical study, *ESAIM*: *COCV* 3 (1998) 163–212.

[2] C. BARDOS, G. LEBEAU and J. RAUCH, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, *SIAM J. Cont. Optim.* 30 (1992) 1024–1065.

[3] N. BURQ, Contrôle de l'équation des ondes dans des domaines peu réguliers, *Asymp. Anal.* 14 (1997) 157–191.

[4] N. BURQ and P. GÉRARD, Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes, *C. R. Acad. Sci. Paris* 325 (1997) 749–752.

[5] R. GLOWINSKI, Ensuring well-posedness by analogy; Stokes problem and boundary control for the wave equation, *J. Compt. Phys.* 103 (2) (1992) 189–221.

[6] R. GLOWINSKI, C.H. LI and J.L. LIONS, A numerical approach to the exact boundary controllability of the wave equation (I). Dirichlet controls: Description of the numerical methods, *Japan. J. Appl. Math.* 7 (1990) 1–76.

[7] R. GLOWINSKI and J.L. LIONS, Exact and approximate controllability for distributed parameter systems, *Acta Numerica* (1996) 159–333.

[8] J.A. INFANTE and E. ZUAZUA, Boundary observability for the space-discretizations of the 1-d wave equation, *C. R. Acad. Sci. Paris, Serie* I 326 (6) (1998) 713–718.

[9] J.A. INFANTE and E. ZUAZUA, Boundary observability for the space semi-discretization of the one-dimensional wave equation, M^2AN, to appear.

[10] A.E. INGHAM, Some trigonometrical inequalities with applications to the theory of series, *Math. Z.* 41 (1936) 367–379.

[11] E. ISAACSON and H.B. KELLER, *Analysis of Numerical Methods*, Wiley, 1966.

[12] V. KOMORNIK, *Exact Controllability and Stabilization. The Multiplier Method*, Willey, 1994.

[13] E.B. LEE and L. MARKUS, *Foundations of Optimal Control Theory*, SIAM Series in Applied Mathematics, Willey, 1967.
[14] J.L. LIONS, *Contrôlabilité Exacte, Stabilisation et Perturbations de Systèmes Distribués. T. 1. Contrôlabilité Exacte*, Masson, RMA 8, Paris, 1988.
[15] J. RALSTON, Gaussian beams and the propagation of singularities, in: W. Littman (Ed.), *Studies in Partial Differential Equations*, MAA Studies in Mathematics, 23, 1982, pp. 206–248.
[16] J. SIMON, Compact sets in the space $L^p(0,T;B)$, *Annali Mat. Pura Appl.* 146 (1987) 65–96.